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ON THE PERIMETER AND THE AREA OF THE CONVEX POLYGONS OF A GIVEN DIAMETER

BY

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1. INTRODUCTION

The regular t-polygons (i.e. polygons with t vertices), have the isoperimetric property, that is, among the t-polygons of a given perimeter only the regular t-polygons have the maximum area (see for instance [2]). Nevertheless, the analogous problem of finding the convex t-polygons of given diameter d which realise the maximum perimeter, gives a new class of convex polygons, the regular (d, t)-polygons (see [4]).

1.1. Definition. *A convex t-polygon P of diameter d is said to be a (d, t)-polygon, if for each edge of P there exists a vertex of P which is at a distance d from the two endpoints of the given edge.*

1.2. Definition. *A regular (d, t)-polygon is a (d, t)-polygon with equal edges.*

However, the existence of regular (d, t)-polygons has been shown only in the case that t has an odd divisor greater than 1. Certainly regular (d, 4)-polygons do not exist. At a recent conference in Israel, D.G. Larman [5] proposed the following problem:

"Let P be a convex polygon in E^2 with diameter 1 and 2^n equal sides. For each side (α, b) there exists a vertex c of P with

$$\|\alpha - c\| = \|b - c\| = 1.$$

Does P exists for $n \geq 3$?"

According to the definitions 1.2 and 1.1. this is equivalent to the problem:

"Do regular $(d, 2^n)$-polygons exist for $n \geq 3$?"

Here, we give a negative answer for $n = 3$.

Also, in this paper we examine the bounds of the maximum perimeter $\ell(t)$ that a convex t-polygon of diameter 1 can get. More precisely for any integer $t \geqslant 3$, we set

$$\ell(t) = \sup\{L(P) : P \text{ is a convex t-polygon of diameter } 1\},$$

where L(P) denotes the perimeter of the polygon P. Then, from [4] in the case that t has an odd divisor greater than 1, we have

$$\ell(t) = 2t \cdot \sin\left(\frac{\pi}{2t}\right)$$

and this value is attained only by the regular (1, t) -polygons. Also we show that

$$\ell(4) = 2 + 2\sqrt{2-\sqrt{3}}$$

and this value is realised only by the (3 | 1, 1, 3) 4-polygon, in terms of the notation that we establish in the §2.

Finally, for $n \geqslant 3$ we have the estimates

$$\ell(2^n) \geqslant \frac{2}{3}(2^n-2)\sin\left[\frac{\pi}{2(2^n-2)}\right] + \frac{4}{3}(2^n+1)\sin\left[\frac{\pi}{2(2^n+1)}\right]$$

in the case that $2^n \equiv 1 \pmod 3$ and

$$\ell(2^n) \geqslant \frac{4}{3}(2^n-1)\sin\left[\frac{\pi}{2(2^n-1)}\right] + \frac{2}{3}(2^n+2)\sin\left[\frac{\pi}{2(2^n+2)}\right],$$

in the case that $2^n \equiv 2 \pmod 3$. These inequalities imply that

$$\ell(2^n) = \pi - \frac{\pi^3}{24 \cdot 2^{2n}} + 0\,(2^{-3n})$$

Also, here we prove that the regular 2^n -polygons of diameter 1 do not realise the maximum perimeter.

Another related problem is to find among the t-polygons of given diameter d those which have the maxumum area.

In this paper we show that if t is an odd integer or if $t = 6$, then, the maximum area is realised only by the regular t-polygon. Also, for $t = 4$, the maximum is attained by all the 4-polygons which have orthogonal diagonals each of length equal to d.

Finally, we remark that if $t = 4n$, for $n \geqslant 1$, there are infinitely many t-polygons which have area equal to the area of the regular t-polygon of diameter d.

2. THE CLASS OF THE (d,t)-POLYGONS

Let P_m denote the regular m-polygon of diameter d, where m is an odd integer, such that $3 \leqslant m \leqslant t$. We construct the arcs of radius d on the sides of P_m exterior to P_m. Next, we consider t-m the number points on these arcs different from the vertices of P_m. Clearly, all these points together with the vertice of P_m are the vertices of a (d, t) -polygon.

We don't know if there exist (d, t)-polygons which are different from these constructed above. Therefore, we give the following definition.

2.1. Definition. *A (d, t)-polygon constructed as abobe on the arcs of radius d, briefly called d- arcs, of the regular m-polygon P_m of diameter d, where m is an odd, with $3 \leq m \leq t$, will be called a (d, m, t) -polygon.*

2.2. Notation. *Les t be an integer with $t \geqslant 3$, and m be an odd integer such that $3 \leq m \leq t$. Let also, $k_1, k_2, ..., k_m$ be m positive integers with*

$$k_1 + k_2 + ... + k_m = t$$

We consider the regular m-polygon P_m of diameter 1. We divide the 1-arcs constructed on the sides of P_m into $k_1, k_2, ..., k_m$ equal arcs, respectively. All the endpoints of these arcs are the vertices of a (l, m, t) -polygon. We shall denote this polygon by $(m \mid k_1, k_2, ..., k_m)$.

In the case that

$$k_1 = k_2 = ... = k_m = k$$

we whall denote simply by $(m \mid k)$.

2.3. Remarks. 1. An $(m \mid k)$-polygon is a regular (1, t)-polygon, where $t = m \cdot k$.

2. If m and n are two different odd divisors of t, then, the regular (1, t)-polygons $\left(m \mid \frac{t}{m} \right)$ and $\left(n \mid \frac{t}{n} \right)$ are different but they have the same perimeter equal to

$$2t \cdot \sin\left(\frac{\pi}{2t} \right)$$

3. If t is an odd number, then, the regular t-polygon of diameter d is a regular (d, t)-polygon. The inverse is not true.

3. THE NON-EXISTENCE OF REGULAR (d, 8)-POLYGONS

Let P be a regular (1, 8)-polygon. From the Lemma 3 and the Theorem 3 of [4] we conclude that P realises the maximum perimeter among the convex 8-polygons of diameter 1. Hence,

$$L(P) = 16 \cdot \sin\left(\frac{\pi}{16}\right)$$

and every side of P has length

$$\alpha = 2 \cdot \sin\left(\frac{\pi}{16}\right).$$

In order to prove that P does not exist, we shall need the following:

3.1. Lemma. *If P is a regular (1,8)-polygon, then, there are three successive vertices of P having distance 1 from the same vertex of P.*

Proof. Let $A_1, A_2, \ldots, A_8$ be the successive vertices of P. For any positive integer i and for any integer j, with $1 \leq j \leq 8$, we put $A_i = A_j$ iff $i \equiv j \pmod 8$. Let also (AB) denote the length of the line segment AB.

We will show that there exist a vertex B of P, such that

$$(BA_i) = (BA_{i+1}) = (BA_{i+2}) = 1,$$

for a positive integer i.

We assume that the above is not true. P is an (1, 8)-polygon, so for the edge A_iA_{i+1} there exists a vertex B of P, such that

$$(BA_i) = (BA_{i+1}) = 1. \tag{1}$$

Also P has equal sides each of length

$$\alpha = 2 \cdot \sin\left(\frac{\pi}{16}\right) < \frac{1}{2}$$

and so the corresponding vertex to the adge A_iA_{i+1} with the property (1) is A_{i+4} or A_{i+5}.

According to the above result, for the edge A_1A_2 the corresponding vertices are A_5 or A_6. We may suppose without loss of the generality that

$$(A_5A_1) = (A_5A_2) = 1 \tag{2}$$

Similary, for the edge A_8A_1 the corresponding vertices are A_4 or A_5 but we must exclude A_5 for otherwise, from (2) we get

$$(A_5A_8) = (A_5A_1) = (A_5A_2) = 1$$

which is a contradiction. Hence,

$$(A_4A_8) = (A_4A_1) = 1 \tag{3}$$

Also, the corresponding vertices of the edge A_5A_6 are A_1 and A_2, but by (2) it must be

$$(A_2A_5) = (A_3A_6) = 1 \tag{4}$$

Next, the corresponding vertices of the edge A_6A_7 are A_2 and A_3, so, by (4) we must have

$$(A_3A_6) = (A_3A_7) = 1 \tag{5}$$

Finally, for the edge A_7A_8 the corresponding vertices are A_3 and A_4. Now, for the vertex A_3 we get from (5) the relation

$$(A_3A_7) = (A_3A_8) = (A_3A_6) = 1$$

and for the vertex A_4 we get from (3) the relation

$$(A_4A_7) = (A_4A_8) = (A_4A_1) = 1$$

which both contradict the assumption that the Lemma is not true. Thus, the result follows.

3.2. Proposition. *Regular (d, 8)-polygons do not exist.*

Proof. Let P be a regular (1,8)-polygon with successive vertices A_1, A_2, ..., A_8. By the lemma 3.1., we may assume that it holds

$$(A_1A_5) = (A_2A_5) = (A_8A_5) = 1 \tag{1}$$

Now, by a result shown in Lemma 3.1., for the edge A_4A_5 either

$$(A_1A_4) = (A_1A_5) = 1 \tag{2}$$

or

$$(A_8A_4) = (A_8A_5) = 1 \tag{3}$$

If the relation (2) is true, then, from (1) we conclude that the points A_1, A_4, A_5, A_8 form a parallegogram with

$$(A_4A_8) > (A_1A_5) = 1$$

which contradicts that the diameter of P is 1. Hence, only the relation (3) is true.

Similarly, for the edge A_5A_6 we have the relation

$$(A_2A_5) = (A_2A_6) = 1 \tag{4}$$

Now the relations (1), (3) and (4) imply that P is symmetrical about

A_1A_5 and the vertices A_8, A_1, A_2, A_4, A_5 and A_6 form a fixed polygon. Also, we have

$$(A_2A_3) = (A_3A_4) = (A_6A_7) = (A_7A_8) = \alpha$$

and so P is fixed.

We have the estimate

$$(A_3A_7) = \frac{\sqrt{2}}{2} \cdot \left[\sqrt{4\left(\frac{\alpha}{\beta}\right)^2 - 1} + 1 \right] \tag{5}$$

where

$$\beta = (A_2A_4) = \sqrt{3 - \sqrt{2} - 2\sqrt{2} \cdot \sin\left(\frac{\pi}{8}\right)} \tag{6}$$

But, form (5) and (6) we get

$$(A_3A_7) \simeq 1.03$$

which gives a contradiction, since the diameter of P is 1. So, regular (d, t)-polygons, for $t = 8$, do not exist.

4. THE MAXIMUM VALUE OF THE PERIMETER OF THE CONVEX POLYGONS

For every convex t-polygon P of diameter 1, from [4] we have

$$L(P) \leqslant 2t \cdot \sin\left(\frac{\pi}{2t}\right)$$

with equality if, and only if P is a regular (1, t)-polygon. Hence, it holds

$$\ell(t) \leq 2t \cdot \sin\left(\frac{\pi}{2\tau}\right)$$

for every $t \geq 3$.

But, in the case that t has an odd divisor greater than 1, the existence of regular (1, t)-polygons has been shown. So, we have

$$\ell(t) = 2t \cdot \sin\left(\frac{\pi}{2t}\right)$$

for every $t = m2^n$, where m is an odd greater than 1 and $n = 0, 1.,$

Also, from the Proposition 3.2., we have the strict inequality

$$L(P) < 16 \cdot \sin\left(\frac{\pi}{16}\right)$$

for every convex 8-polygon P of diameteer 1. Therefore, even if

$$\ell(8) = 16 = \sin\left(\frac{\pi}{16}\right)$$

this value is not attained by any convex 8-polygon of diameter 1.

In order to find $\ell(4)$ we prove the following:

4.1. Proposition. *Among the convex 4-polygons of diameter 1, the maximum perimeter has only the $(3 \mid 1, 1, 2)$ 4-polygon.*

Proof. Let A, B, C and D be the vertices of a convex quadrilateral P of diameter 1 and maximal perimeter. Let also AC be a diagonal of P of length 1.

We put

$$2\alpha_1 = (AB) + (BC)$$

and

$$2\alpha_2 = (AD) + (DC)$$

Assuming that α_1 is constant, the vertex B is a point of the ellipse with foci the points A, C and sum $2\alpha_1$. Similarly, D is a point of the ellipse with foci the points A, C and sum $2\alpha_2$. Let AB be the side of P with the greatest length. From the maximality of $\alpha_1 + \alpha_2$, applying elementary geometrical methods, if follows that BD is vertical to AC and

$$(AB) = (BD) = 1$$

Let now ϕ be the angle CAB, We may suppose that $\phi \geqslant \frac{\pi}{4}$, for otherwise we set ϕ to be the angle DBA. Since the diameter of P is 1, we have $(BC) \leqslant 1$ and this implies that $\phi \leqslant \frac{\pi}{3}$. Hence,

$$\frac{\pi}{4} \leqslant \phi \leqslant \frac{\pi}{3}.$$

Then, we get

$$L(P) = (AB) + (BC) + (CD) + (DA) =$$

$$= 1 + 2\sin\left(\frac{\phi}{2}\right) + \sqrt{3 - 2\sin\phi - 2\cos\phi} + 2\sin\left(\frac{\pi}{4} - \frac{\phi}{2}\right)$$

and so

$$\frac{d}{d\phi} L(P) = \cos\left(\frac{\phi}{2}\right) + \frac{\sin\phi - \cos\phi}{\sqrt{3 - 2\sin\phi - 2\cos\phi}} - \cos\left(\frac{\pi}{4} - \frac{\phi}{2}\right).$$

Now, since $\phi \geq \frac{\pi}{4}$ the strict inequality

$$\frac{d}{d\phi} L(P) > 0$$

is equivalent to the inequality

$$\sqrt{2}\cos\left(\phi - \frac{\pi}{4}\right) + \frac{3\sqrt{2} - 2}{4} > 0$$

which is true for every $\phi \in \left[\frac{\pi}{4}, \frac{\pi}{3}\right]$.

Hence, $L(P)$ is a strictly increasing function of ϕ in $\left[\frac{\pi}{4}, \frac{\pi}{3}\right]$, which implies that for every 4-polygon P it holds

$$L(P) \leqslant 2 + 4\sin\left(\frac{\pi}{12}\right) = 2 + 2\sqrt{2 - \sqrt{3}}$$

with equality if, and only if, P is a $(3 \mid 1, 1, 2)$ 4-polygon.

4.2. Corollary. $\ell(4) = 2 + 2\sqrt{2 - \sqrt{3}}$,

and this value is realised only by the $(3 \mid 1, 1, 2)$ 4-polygon.

Proof. This is an immediate consequence of the Proposition 4.1.

Now we prove a result concerning the $(1, m, 2^n)$-polygons, according to the definition 2.1.

4.3. Theorem. *For $n \geq 3$, among the $(1, m, 2^n)$-polygons, the maximum perimeter has only the $(3 \mid k, k, k+1)$ 2^n-polygons, the case that $2^n = 3k + 1$, or the $(3 \mid k, k+1, k+1)$ 2^n-polygon in the case that $2^n = 3k + 2$.*

First we prove some Lemmas.

4.3.1. Lemma. *Let $f(x)$ be a strictly concave real function. If α, β, γ and δ are in the domain of the function $f(x)$, such that*

$$\alpha + \beta = \gamma + \delta$$

and $\alpha < \gamma \leqslant \delta < \beta$, *then*

$$f(\alpha) + f(\beta) < f(\gamma) + f(\delta).$$

Proof. Since f(x) is a strictly concave function it follows that f′(x) is strictly decreasing function and hence, from the mean value theorem, we have

$$\frac{f(\gamma) - f(\alpha)}{\gamma - \alpha} > \frac{f(\beta) - f(\delta)}{\beta - \delta}.$$

But $\gamma - \alpha = \beta - \delta > 0$, and so we get the result.

4.3.2. Lemma. *Let* t *and* m *be two given integers, such that* $3 \leqslant m \leqslant t$, *and* m *odd. If* P *is an* (1, m, t)*-polygon of maximum perimeter, then,* P *is an* $(m \mid k_1, k_2, \ldots, k_m)$ *t-polygon, with*

$$|k_i - k_j| \leqslant 1,$$

for every $i, j = 1, 2, \ldots, m$.

Proof. Let P* denote the union of the 1-arcs consttructed on the sides of P, exterior to P. P is an (1, m, t)-polygon, and so P* consists of m the number equal arcs each of length $\frac{\pi}{m}$.

Let $A_0, A_1, \ldots, A_r$ be the successive vertices of P which belong to one of these arcs and ϕ_i be the length of the arc $A_{i-1}A_i$, $i = 1, \ldots, r$.

The function f(x) = sinx is strictly concave in $\left(0, \frac{\pi}{2}\right)$ and since

$$\phi_1 + \phi_2 + \ldots + \phi_r = \frac{\pi}{m},$$

we have that

$$\sum_{i=0}^{r-1} (A_iA_{i+1}) = 2 \sum_{i=1}^{r} \sin\left(\frac{\phi_i}{2}\right) \leqslant 2r \cdot \sin\left(\frac{\pi}{2rm}\right),$$

with equality if, and only if,

$$\phi_1 = \phi_2 = \ldots = \phi_r = \frac{\pi}{rm}.$$

This implies that each of the m arcs of P* is divided into equal parts for otherwise, P would not be of maximal perimeter. Thus, P is an $(m \mid k_1, k_2, \ldots, k_m)$ t-polygon, with perimeter

$$L(P) = \sum_{i=1}^{m} 2k_i \cdot \sin\left(\frac{\pi}{2mk_i}\right) \tag{1}$$

Now, the function $f(x) = x\sin\dfrac{\pi}{2mx}$ is strictly concave in $[1, +\infty)$ because

$$f''(x) = -\frac{1}{x^3}\left(\frac{\pi}{2m}\right)^2 \cdot \sin\left(\frac{\pi}{2mx}\right) < 0,$$

for every $x \geq 1$.

If we assume that for two positive integers it holds

$$k_i - k_j \geqslant 2,$$

we set $k_i' = k_i - 1$ and $k_j' = k_j + 1$. So, applying the Lemma 4.3.1. we get

$$k_i\sin\left(\frac{\pi}{2mk_i}\right) + k_j\sin\left(\frac{\pi}{2mk_j}\right) < k_i'\sin\left(\frac{\pi}{2mk_i'}\right) + k_j'\sin\left(\frac{\pi}{2mk_j'}\right) \quad (2)$$

The relations (1) and (2) contradict the maximality of $L(P)$, hence, for every $i, j = 1, 2, ..., m$ we have

$$|k_i - k_j| \leqslant 1$$

4.3.3. Lemma. *Let t, m and P be as in the Lemma 4.3.2. If m is not a divisor of t, then, P is an $(m \mid k, k, ..., k, k+1, k+1, ..., k+1)$ t-polygon.*

Proof. From the Lemma 4.3.2, we have that P is an $(m \mid k_1, k_2, ..., k_m)$ t-polygon, with

$$|k_i - k_j| \leqslant 1$$

for every $i, j = 1, 2, ..., m$.

The integers k_i, $i = 1, 2, ..., m$, are not all equal, for otherwise m would be a divisor of t, Hence, there exist $i, j \in \{1, 2, ..., m\}$ such that $k_j < k_j$. We put $k_i = k$, then $k_j = k + 1$.

Thus, for any $r = 1, 2, ..., m$, with $r \neq i,j$, $k_r = k$ or $k_r = k + 1$ and this implies the result.

4.3.4. Remark. Given the positive integers t and m as in the previous Lemma, there exist unique positive integers k and υ, such that

$$t = km + \upsilon, \quad 0 < \upsilon < m$$

If $1 \leqslant \upsilon \leqslant m-2$, then

$$mk \leqslant mk + \upsilon - 1 < mk + \upsilon + 2 \leqslant mk + m$$

The function $f(x)$ is strictly concave, so, we have the relations

$$\frac{m-\upsilon}{m}\cdot f(mk) + \frac{\upsilon}{m}\cdot f(mk+m) \leqslant \frac{2}{\upsilon+2}\cdot f(mk) + \frac{\upsilon}{\upsilon+2}\cdot f(mk+\upsilon+2) \quad (3)$$

$$\frac{2}{\upsilon+2}\cdot f(mk) + \frac{\upsilon}{\upsilon+2}\cdot f(mk+\upsilon+2) \leqslant \frac{2}{3}\cdot f(mk+\upsilon-1) + \frac{1}{3}\cdot f(mk+\upsilon+2) \quad (4)$$

In (3) the equality holds exactly when $\upsilon = m-2$ and in (4) exactly when $\upsilon = 1$, hence, the inequality (2) is true.

If $\upsilon = m-1$, then, we put $c = 2^n + 1 = mk + m$, and the relation (2) becomes

$$\frac{1}{m}\cdot f(c-m) + \frac{m-1}{m}\cdot f(c) < \frac{2}{3}\cdot f(c-2) + \frac{1}{3}\cdot f(c+1) \quad (5)$$

Next, we consider the function

$$g(m) = \frac{1}{m}\cdot f(c-m) + \frac{m-1}{m}\cdot f(c) \quad (6)$$

and we have

$$g'(m) = \frac{1}{m}\left[\frac{f(c)-f(c-m)}{m} - f'(c-m)\right].$$

Since the function $f'(x)$ is strictly decreasing, by the mean value theorem, it follows that $g'(m) < 0$, and so the function $g(m)$ is strictly decreasing. Thus, for every $m \geq 5$, we have

$$g(m) < g(3) = \frac{1}{3}\cdot f(c-3) + \frac{2}{3}\cdot f(c) \quad (7)$$

Now, the relation

$$\frac{1}{3} f(c-3) + \frac{2}{3} f(c) < \frac{2}{3} f(c-2) + \frac{1}{3} f(c+1) \quad (8)$$

is equivalent to the inequality

$$\frac{f(c)-f(c-2)}{2} < \frac{f(c+1)-f(c-3)}{4},$$

which is true since the function $f(x)$ is strictly increasing and strictly concave (see for example [1] page 47).

Hence, from the relations (6), (7) and (8), it follows that the realtion (5) is true. But this implies that the relation (2) is true and so P is a $(3 \mid k, k, k+1)$ 2^n-polygon (in the case that $2^n = 1 \pmod 3$).

CASE II : n is odd. In this case $2^n = 3p + 2$, for some integer $p \geqslant 2$. As in the case I, it suffices to show that

$$\frac{m - \upsilon}{m} f(mk) + \frac{\upsilon}{m} f(mk + m) < \frac{2}{3} f(3p + 3) + \frac{1}{3} f(3p)$$

for $m \geqslant 5$. But $3p + 2 = mk + \upsilon$, so the above is equivalent to

$$\frac{m - \upsilon}{m} f(mk) + \frac{\upsilon}{m} f(mk + m) < \frac{2}{3} f(mk + \upsilon + 1) + \frac{1}{3} f(mk + \upsilon - 2) \quad (9)$$

If $2 \leqslant \upsilon \leqslant m - 1$, then,

$$mk \leqslant mk + \upsilon - 2 < mk + \upsilon + 1 \leqslant mk + m$$

and since the function $f(x)$ is strictly concave, we have the relations

$$\frac{m - \upsilon}{m} f(mk) + \frac{\upsilon}{m} f(mk + m) \leqslant \frac{\upsilon}{\upsilon + 1} f(mk + \upsilon + 1) + \frac{1}{\upsilon + 1} f(mk) \quad (10)$$

$$\frac{\upsilon}{\upsilon + 1} f(mk + \upsilon + 1) + \frac{1}{\upsilon + 1} f(mk) \leqslant \frac{2}{3} f(mk + \upsilon + 1) + \frac{1}{3} f(mk + \upsilon - 2) \quad (11)$$

The equality holds in (10) exactly when $\upsilon = m - 1$, and in (11) exactly $\upsilon = 2$. Hence, the strict inequality (9) is true.

If $\upsilon = 1$, then, we put $c = 2^n - 1 = mk$, and the relation (9) becomes

$$\frac{m - 1}{m} f(c) + \frac{1}{m} f(c + m) < \frac{2}{3} f(c + 2) + \frac{1}{3} f(c - 1) \quad (12)$$

Now, we consider the function

$$g(m) = \frac{m - 1}{m} f(c) + \frac{1}{m} f(c + m) \quad (13)$$

which has first derivative

$$g'(m) = -\frac{1}{m}\left[\frac{f(c + m) - f(c)}{m} - f'(c + m)\right] < 0$$

as in the case I. Thus, the function $g(m)$ is strictly decreasing and since $m \geqslant 5$, we have

$$g(m) < g(3) = \frac{2}{3} f(c) + \frac{1}{3} f(c+3) \tag{14}$$

Finally, since the function $f(x)$ is strictly increasing and strictly concave, it holds

$$\frac{f(c+3) - f(c-1)}{4} < \frac{f(c+2) - f(c)}{2} \tag{15}$$

The relations (13), (14) and (15) prove the ineguality (12), and so the relation (9) is true.

4.4. Proposition. *The regular 2^n-polygon of diameter 1 has perimeter strictly less than the perimeter of the $(3 \mid k, k, k+1)$ 2^n-polygon in the case that $2^n = 3k+1$, or the perimeter of the $(3 \mid k+1, k+1, k)$ 2^n-polygon in the case that $2^n = 3k+2$.*

Proof. The perimeter of the regular 2^n-polygon of diameter 1 is equal to

$$2^n \sin\left(\frac{\pi}{2^n}\right),$$

We consider the function

$$f(x) = x\sin\left(\frac{\pi}{2x}\right)$$

then, it is sufficient to show that

$$f\left(\frac{3k+1}{2}\right) < \frac{2}{3} f(3k) + \frac{1}{3} f(3k+3),$$

in the case that $2^n = 3k+1$, and

$$f\left(\frac{3k+2}{2}\right) < \frac{2}{3} f(3k+3) + \frac{1}{3} f(3k),$$

in the case that $2^n = 3k+2$. But, these relations are obvious since the function $f(x)$ is strictly increasing (see the proof of the theorem 4.4) and $k \geqslant 2$.

4.5. Remark. Considering the $(3 \mid k, k, k+1)$ or the $(3 \mid k+1, k+1, k)$ 2^n-polygons, we conclude that

$$\ell(2^n) \geqslant 4k\sin\left(\frac{\pi}{6k}\right) + 2(k+1)\sin\left(\frac{\pi}{6(\kappa+1)}\right)$$

in the case that $2^n = 3k+1$, and

$$\ell(2^n) \geqslant 2k\sin\left(\frac{\pi}{6k}\right) + 4(k+1)\sin\left(\frac{\pi}{6(\kappa+1)}\right)$$

in the case that $2^n = 3k+2$.

Now, we quote the following:

Problem: *"Does there exist any convex 2^n-polygon of diameter 1 which have perimeter strictly greater than the perimeter of the $(3 \mid k, k, k+1)$ or the $(3 \mid k+1, k+1, k)$ 2^n-polygon, for $n \geqslant 3$?". (For $n = 2$ does not exist any).*

According to the theorem 4.3. the possible existence of such a polygon must be searched outside the class of the $(1, m, 2^n)$-polygons.

An immediate consequence of the remark 4.5 and the inequality

$$\ell(2^n) \leqslant 2t \cdot \sin\left(\frac{\pi}{2t}\right)$$

for $t \geqslant 3$, is the following:

4.6. Theorem. $\ell(2^n) = \pi - \dfrac{\pi^3}{24 \cdot 2^{2n}} + O(2^{-3n}).$

5. THE LARGEST PERIMETER OF THE CONVEX POLYGONS WITH EQUAL SIDES

A well known problem (see for example [3]) is to find the number $P_2(t)$ defined as the least positive number such that any convex domain of diameter 1, can have its boundary divided into t sets, each of diameter at most $P_2(t)$. A closely related problem is to find the largest possible perimeter of a convex t-polygon of given diameter. Furthermore, we may restrict this problem in the class of the convex t-polygons with equal sides. In the case that t has an odd divisor strictly greater than 1, the answer is the square of diameter 1. However, if $t = 2^n$, with $n \geqslant 3$, the problem remains open.

Here, we make the following conjecture:

"Among the convex 2^n-polygons of diameter 1 and equal sides, only the regular polygons realise the maximum perimeter".

6. THE POLYGONS OF A GIVEN DIAMETER WITH THE GREATEST AREA

First we prove the following:

6.1. Theorem. *Among the t-polygons of diameter d, where t is an odd, only the regular t-polygons have the maximum area.*

Proof. Let P be a t-polygon of diameter 1 and of maximum area. Certainly, P is a convex t-polygon.

Now, for a given perimeter the maxumum area is realised only by the regular t-polygons. Also, since t is an odd integer, from [4] among the convex t-polygons of diameter 1, only the regular (1, t)-polygons have the maximum perimeter. But a regular t-polygon of diameter 1, since t is odd, is also a regular (1, t)-polygon. Hence, the polygon P is of maximum area if, and only if, it is regular.

Next, for the cases $t = 4$ and $t = 6$, we whall need a Lemma the proof of which is obvious.

6.2. Lemma. *Let Q be a quadrilateral with diagonals of length α and β respectively. Then, Q has maximum area $\frac{\alpha\beta}{2}$ if, and only if Q has orthogonal diagonals.*

6.3. Corollary. Among the 4-polygons of diameter 1, the maximum area realise all the 4-polygons with orthogonal diagonals each of length 1.

6.4. Proposition. *Among the hexagons of diameter 1, only the regular hexagon has maximum area.*

Proof. Let A_i, $i = 1, 2, ..., 6$, be the successive vertices of the hexagon P of diameter 1.

Let x, y, z be the lengths of the line segments A_1A_3, A_3A_5 and A_5A_1 respectively. Let also E_x, E_y and E_z denote the areas of the quadrilarals $A_1A_2A_3A_5$, $A_4A_5A_1A_3$ and $A_5A_6A_1A_3$ respectively.

From the Lemma 6.2 we have the relations

$$E_x \leqslant \frac{x}{2}, \quad E_y \leqslant \frac{y}{2}, \quad E_z \leqslant \frac{z}{2} \tag{1}$$

If E is the area of P and S is the area of the triangle $A_1A_3A_5$ then, we have from (1)

$$E = E_x + E_y + E_z - S \leqslant \frac{x+y+z}{2} + 2S,$$

with equality if, and only if A_1A_4, A_2A_5, A_3A_6 are orthogonal to A_3A_5, A_1A_3, A_1A_5 respectively. But

$$S = \sqrt{\tau(\tau - x)(\tau - y)(\tau - z)}$$

where $\tau = \frac{x+y+z}{2}$. Hence, considering the function $f(x, y, z) = \frac{\tau}{2} - S$ and differentiating, we find that it gets its maximum value exactly when

$$x = y = z = \frac{\sqrt{3}}{2} \tag{2}$$

Thus,

$$E \leqslant \frac{3\sqrt{3}}{8}$$

with equality if, and only if, A_1A_4, A_2A_5, A_3A_6 are orthogonal to A_3A_5, A_1A_3, A_1A_5 respectively, and the relation (2) holds. But this implies that the hexagon P has the maximum area exactly when it is regular.

6.5 Remark. If P is a regular 4n-polygon of diameter 1, then there are ifninitely many 4n-polygons of diameter 1 and the area of P, since we may remove a diagonal for example without increasing the diameter of P and without changing the area.

So we make the following conjectures:

Conjecture 1. *Among the t-polygons of diameter 1, the regular t-polygons realise the maximum area.*

Conjecture 2. *Let n be an odd integer, $n > 3$, then among the 2n-polygons of diameter 1, only the regular 2n-polygons realise the maximum area.*

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