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TOPOLOGICAL GROUPS AND K-SOUSLIN SETS

BY

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In this note we are concerned with topological groups, in relation with K-Souslin sets and in particular with Souslin sets. We know that every Souslin space is a Radon space, i.e. on these spaces every finite Borel measure is a Radon measure ([8]). Using the concept of topological group we state the definition of a topological action. When we consider topological action in a Souslin space we prove new topological properties for the topological group, (Th. 2.1.) and for the topological space (Corr. 2.4.) if the group is Polish.

In particular, the measurability of Radon spaces can be transferred from topological space to the topological group (Th. 2.1.) and vice-versa (Corr. 2.4.).

In Prop. 3.2. we state a proposition which constitutes an extended form of [5, 9.15]. For basic definitions and theorems on K-Souslin sets see: [1, Chap. 3], [3, Ch. IX], [4, 1], [8, Ch. II], [9], and for definitions and theorems on topological actions see: [4, Ch. III, §4], [6, II, Ch. 12].

1. Notations and preliminaries

1.1. A set G endowed with the algebraic structures of a group and a (compatible) topology such that the map

$$G x G \to G: (x, y) \to xy^{-1}$$

to be jointly continous, is said to be topological group.

1.2. An algebraic action is a trial (G, E, f) where G is a group, $E \neq \emptyset$ and f: $G x E \to E : f(g, x) \equiv gx$ is the action map with the properties:

$$\text{(i) } f(e, x) = x, \quad \text{(ii) } f(g_1, f(g_2, x)) = f(g_1 g_2, x)$$

A topological action is an algebraic one, where G is a topological group, the space E is a topological space and the map f is continous ([3, 5]).

A group G acts freely in a set E if by the relation

$$g \cdot x = x, \quad x \in E, \quad g \in G \quad \text{we have} \quad g = e$$

$$h_x : G \to E: s \to s \cdot x$$

we have the following decompositions:

$$h_x : G \xrightarrow{\pi_x} G/S_x \xrightarrow{f_x} E$$

where π_x, is the canonical map, $S_x = \{ s \in G : s \cdot x = x, \ x \in E\}$ is the stability group, and f_x is a bijection map ([6, 11]).

Corrollary 2.4. *Let G be a compact and metrisable topological group which acts continuously and transitively in a topological space E. Then, h_x is open, continuous, onto and semi-proper ([7]). If the space is Hausdorff is polish.*

Proof: Obvious by [6, II, p. 46, p. 54], [7, p. 5-02, Lemme 3], [8, Corr. 2, p. 106].

3. Measures on polish groups

3.1. If μ_1, μ_2 are measures on the measurable space (X, A) we have $\mu_1 << \mu_2$ (μ_1 is dominated by μ_2) iff $\forall E \in A$ with $\mu_2(E) = 0$, then $\mu_1(E) = 0$([2, p. 156]).

Proposition 3.2. *Let G be a locally compact metrisable topological group and H be a subgroup of G open and σ-compact. If $f \in K(H)$ (K(H) is the set of continuous functions with compact support) then, for Borel measure μ such that.*
$\mu << \mu_1$, where μ_1 is a finite Borel measure on H, the maps

$$x \to \int f(xy)\mu(dy)$$
$$x \to \int f(yx)\mu(dy) \qquad x \in H$$

are continuous.

Proof: Since G is locally compact group there is a polish subgroup H([5, p. 301], [3, Ch. IX, Prop. 2]). The measure μ_1 is a Radon regular measure ([8], [5, p. 294]). By the relation $\mu << \mu_1$ and since μ_1 is regular, μ is regular ([2, p. 199]) Borel measure. Then, ([5, p. 300]), the maps are continuous.

3.3 Let G be a locally compact topological group and $\mu \neq 0$ Borel regular measure on G.

The topological group acts properly in E if the map

∂: GxE → ExE: $\partial(g, x) = (x, gx)$ is proper ([3, Ch. I, §10, Ch III, §4]).

G acts transitively in E, if for every couple of elements x, y of E there exists $s \in G$ such that $y = s \cdot x$.

G acts continously in E, if the application

$(s, x) \rightarrow s \cdot x.$, GxE → E is continous.

2. On topological action

Theorem 2.1. *Let E be a polish space and G be a Hausdorff topological group which acts properly and freely in E. Then, G is a Souslin space.*

Proof: The map ∂: GxE → ExE: $(g, x) \rightarrow (x, g \cdot x)$ is proper. Also, it is $1 - 1$. In fact, if $\partial(g, x) = \partial(g', x')$ we have $(x, g \cdot x) = (x', g' x')$ i.e. $x = x'$ and $g \cdot x = g'x' = g'x$.

Since G acts freely, $g = g'$ hence $(g, x) = (g', x')$. The map ∂ is proper, thus the set $\partial(GxE)$ is closed ([3, Ch. I, §10, Pr. 2]) and the restriction of ϑ^{-1} on closed subset $\{x\} \times E$ of $\partial(GxE)$ defines the map

$$\partial^{-1} : \{x\}\, xE\, \{x\{ : (x, g \cdot x) \rightarrow (g, x)$$

which is continous ([3, 1, §10, Prop. 5, III, §4, Prop. 4]). By assumption E is a polish space and thus Gx{x} is Souslin. Since the projection $pr_1 : Gx\{x\} \rightarrow G$ is continuous, we have the result of the theorem.

2.2. Let (G, E, f) and (G′, E′, f′) be two actions. A map $\psi : E \rightarrow E'$ is said to be compatible with a group homomorphism $g : G \rightarrow G$: if $\psi(s \cdot x) = g(s)\, \psi(x)$, $s \in G$, $x \in E$.

Corrollary 2.3. *Let (G, E, f), (G′, E′, f′) be two topological actions where E and E′ are polish spaces. Let $g : G \rightarrow G'$ be a proper, $1 - 1$, morphism of topological group and $\psi : E \rightarrow G'$ be a continuous map. If the map ψ is compatible with g and (G′, E′, f′) satisfies the assumptions of the previous theorem, then G is a Souslin space.*

Proof: [4, III, §4, Pr. 5, Def. 2].

Let G be a topological group which acts continuously and rransitively on a topological space E. For the continuous map

The measure μ is a Haar measure if $\mu(xA) = \mu(A)$ $x \in G, A \in B(G)$ where $B(G)$ are the Borel sets on G. The μ is a right Haar measure if $\mu(Ax) = \mu(A)$ $x \in G$, $A \in B(G)$. Immediately, if $A \in B(G)$, then Ax, xA belong to $B(G)$.

Proposition 3.4. *Let G be a locally compact topological group.*

(i) If G is polish space and μ is a left Haar measure on G, the μ is σ-finite.

(ii) If G is a metrisable topological group and if μ is a σ-finite left Haar measure on an open subgroup H of G, then H is a polish subgroup.

Proof: (i) Since G is locally compact polish space is immediately Lindelöf topological group and thus G is σ-compact. Thus ([5, p. 317]) the left Haap measure is σ-finite.

(ii) Since the Haap measure is σ-finite, then H is σ-compact ([5, p. 301, p. 317]). Now, H by [4, Ch. IX, Prop. 2] is a polish subgroup.

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ON THE PERIMETER AND THE AREA OF THE CONVEX POLYGONS OF A GIVEN DIAMETER

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1. INTRODUCTION

The regular t-polygons (i.e. polygons with t vertices), have the isoperimetric property, that is, among the t-polygons of a given perimeter only the regular t-polygons have the maximum area (see for instance [2]). Nevertheless, the analogous problem of finding the convex t-polygons of given diameter d which realise the maximum perimeter, gives a new class of convex polygons, the regular (d, t)-polygons (see [4]).

1.1. Definition. *A convex t-polygon P of diameter d is said to be a (d, t)-polygon, if for each edge of P there exists a vertex of P which is at a distance d from the two endpoints of the given edge.*

1.2. Definition. *A regular (d, t)-polygon is a (d, t)-polygon with equal edges.*

However, the existence of regular (d, t)-polygons has been shown only in the case that t has an odd divisor greater than 1. Certainly regular $(d, 4)$-polygons do not exist. At a recent conference in Israel, D.G. Larman [5] proposed the following problem:

"Let P be a convex polygon in E^2 with diameter 1 and 2^n equal sides. For each side (α, b) there exists a vertex c of P with

$$\|\alpha - c\| = \|b - c\| = 1.$$

Does P exists for $n \geq 3$?"

According to the definitions 1.2 and 1.1. this is equivalent to the problem:

"Do regular $(d, 2^n)$-polygons exist for $n \geq 3$?"

Here, we give a negative answer for $n = 3$.

Also, in this paper we examine the bounds of the maximum perimeter $\ell(t)$ that a convex t-polygon of diameter 1 can get. More precisely for any integer $t \geqslant 3$, we set

$$\ell(t) = \sup\{L(P) : P \text{ is a convex t-polygon of diameter } 1\},$$

where L(P) denotes the perimeter of the polygon P. Then, from [4] in the case that t has an odd divisor greater than 1, we have

$$\ell(t) = 2t \cdot \sin\left(\frac{\pi}{2t}\right)$$

and this value is attained only by the regular (1, t) -polygons. Also we show that

$$\ell(4) = 2 + 2\sqrt{2 - \sqrt{3}}$$

and this value is realised only by the (3 | 1, 1, 3) 4-polygon, in terms of the notation that we establish in the §2.

Finally, for $n \geqslant 3$ we have the estimates

$$\ell(2^n) \geqslant \frac{2}{3}(2^n - 2)\sin\left[\frac{\pi}{2(2^n - 2)}\right] + \frac{4}{3}(2^n + 1)\sin\left[\frac{\pi}{2(2^n + 1)}\right]$$

in the case that $2^n \equiv 1 \pmod 3$ and

$$\ell(2^n) \geqslant \frac{4}{3}(2^n - 1)\sin\left[\frac{\pi}{2(2^n - 1)}\right] + \frac{2}{3}(2^n + 2)\sin\left[\frac{\pi}{2(2^n + 2)}\right],$$

in the case that $2^n \equiv 2 \pmod 3$. These inequalities imply that

$$\ell(2^n) = \pi - \frac{\pi^3}{24 \cdot 2^{2n}} + 0\,(2^{-3n})$$

Also, here we prove that the regular 2^n -polygons of diameter 1 do not realise the maximum perimeter.

Another related problem is to find among the t-polygons of given diameter d those which have the maxumum area.

In this paper we show that if t is an odd integer or if $t = 6$, then, the maximum area is realised only by the regular t-polygon. Also, for $t = 4$, the maximum is attained by all the 4-polygons which have orthogonal diagonals each of length equal to d.

Finally, we remark that if $t = 4n$, for $n \geqslant 1$, there are infinitely many t-polygons which have area equal to the area of the regular t-polygon of diameter d.

2. THE CLASS OF THE (d,t)-POLYGONS

Let P_m denote the regular m-polygon of diameter d, where m is an odd integer, such that $3 \leqslant m \leqslant t$. We construct the arcs of radius d on the sides of P_m exterior to P_m. Next, we consider t-m the number points on these arcs different from the vertices of P_m. Clearly, all these points together with the vertice of P_m are the vertices of a (d, t) -polygon.

We don't know if there exist (d, t)-polygons which are different from these constructed above. Therefore, we give the following definition.

2.1. Definition. *A (d, t)-polygon constructed as abobe on the arcs of radius d, briefly called d- arcs, of the regular m-polygon P_m of diameter d, where m is an odd, with $3 \leq m \leq t$, will be called a (d, m, t) -polygon.*

2.2. Notation. *Les t be an integer with $t \geqslant 3$, and m be an odd integer such that $3 \leq m \leq t$. Let also, $k_1, k_2, \ldots, k_m$ be m positive integers with*

$$k_1 + k_2 + \ldots + k_m = t$$

We consider the regular m-polygon P_m of diameter 1. We divide the 1-arcs constructed on the sides of P_m into $k_1, k_2, \ldots, k_m$ equal arcs, respectively. All the endpoints of these arcs are the vertices of a (l, m, t) -polygon. We shall denote this polygon by $(m \mid k_1, k_2, \ldots, k_m)$.

In the case that

$$k_1 = k_2 = \ldots = k_m = k$$

we whall denote simply by $(m \mid k)$.

2.3. Remarks. 1. An $(m \mid k)$-polygon is a regular (1, t)-polygon, where $t = m \cdot k$.

2. If m and n are two different odd divisors of t, then, the regular (1, t)-polygons $\left(m \mid \dfrac{t}{m}\right)$ and $\left(n \mid \dfrac{t}{n}\right)$ are different but they have the same perimeter equal to

$$2t \cdot \sin\left(\frac{\pi}{2t}\right)$$

3. If t is an odd number, then, the regular t-polygon of diameter d is a regular (d, t)-polygon. The inverse is not true.

3. THE NON-EXISTENCE OF REGULAR (d, 8)-POLYGONS

Let P be a regular (1, 8)-polygon. From the Lemma 3 and the Theorem 3 of [4] we conclude that P realises the maximum perimeter among the convex 8-polygons of diameter 1. Hence,

$$L(P) = 16 \cdot \sin\left(\frac{\pi}{16}\right)$$

and every side of P has length

$$\alpha = 2 \cdot \sin\left(\frac{\pi}{16}\right).$$

In order to prove that P does not exist, we shall need the following:

3.1. Lemma. *If P is a regular (1,8)-polygon, then, there are three successive vertices of P having distance 1 from the same vertex of P.*

Proof. Let $A_1, A_2, \ldots, A_8$ be the successive vertices of P. For any positive integer i and for any integer j, with $1 \leq j \leq 8$, we put $A_i = A_j$ iff $i \equiv j \pmod 8$. Let also (AB) denote the length of the line segment AB.

We will show that there exist a vertex B of P, such that

$$(BA_i) = (BA_{i+1}) = (BA_{i+2}) = 1,$$

for a positive integer i.

We assume that the above is not true. P is an (1, 8)-polygon, so for the edge A_iA_{i+1} there exists a vertex B of P, such that

$$(BA_i) = (BA_{i+1}) = 1. \tag{1}$$

Also P has equal sides each of length

$$\alpha = 2 \cdot \sin\left(\frac{\pi}{16}\right) < \frac{1}{2}$$

and so the corresponding vertex to the adge A_iA_{i+1} with the property (1) is A_{i+4} or A_{i+5}.

According to the above result, for the edge A_1A_2 the corresponding vertices are A_5 or A_6. We may suppose without loss of the generality that

$$(A_5A_1) = (A_5A_2) = 1 \tag{2}$$

Similary, for the edge A_8A_1 the corresponding vertices are A_4 or A_5 but we must exclude A_5 for otherwise, from (2) we get

$$(A_5A_8) = (A_5A_1) = (A_5A_2) = 1$$

which is a contradiction. Hence,

$$(A_4A_8) = (A_4A_1) = 1 \tag{3}$$

Also, the corresponding vertices of the edge A_5A_6 are A_1 and A_2, but by (2) it must be

$$(A_2A_5) = (A_3A_6) = 1 \tag{4}$$

Next, the corresponding vertices of the edge A_6A_7 are A_2 and A_3, so, by (4) we must have

$$(A_3A_6) = (A_3A_7) = 1 \tag{5}$$

Finally, for the edge A_7A_8 the corresponding vertices are A_3 and A_4. Now, for the vertex A_3 we get from (5) the relation

$$(A_3A_7) = (A_3A_8) = (A_3A_6) = 1$$

and for the vertex A_4 we get from (3) the relation

$$(A_4A_7) = (A_4A_8) = (A_4A_1) = 1$$

which both contradict the assumption that the Lemma is not true. Thus, the result follows.

3.2. Proposition. *Regular (d, 8)-polygons do not exist.*

Proof. Let P be a regular (1,8)-polygon with successive vertices A_1, A_2, ..., A_8. By the lemma 3.1., we may assume that it holds

$$(A_1A_5) = (A_2A_5) = (A_8A_5) = 1 \tag{1}$$

Now, by a result shown in Lemma 3.1., for the edge A_4A_5 either

$$(A_1A_4) = (A_1A_5) = 1 \tag{2}$$

or

$$(A_8A_4) = (A_8A_5) = 1 \tag{3}$$

If the relation (2) is true, then, from (1) we conclude that the points A_1, A_4, A_5, A_8 form a parallegogram with

$$(A_4A_8) > (A_1A_5) = 1$$

which contradicts that the diameter of P is 1. Hence, only the relation (3) is true.

Similarly, for the edge A_5A_6 we have the relation

$$(A_2A_5) = (A_2A_6) = 1 \tag{4}$$

Now the relations (1), (3) and (4) imply that P is symmetrical about

A_1A_5 and the vertices A_8, A_1, A_2, A_4, A_5 and A_6 form a fixed polygon. Also, we have

$$(A_2A_3) = (A_3A_4) = (A_6A_7) = (A_7A_8) = \alpha$$

and so P is fixed.

We have the estimate

$$(A_3A_7) = \frac{\sqrt{2}}{2}\cdot\left[\sqrt{4\left(\frac{\alpha}{\beta}\right)^2 - 1} + 1\right] \tag{5}$$

where

$$\beta = (A_2A_4) = \sqrt{3 - \sqrt{2} - 2\sqrt{2}\cdot\sin\left(\frac{\pi}{8}\right)} \tag{6}$$

But, form (5) and (6) we get

$$(A_3A_7) \simeq 1.03$$

which gives a contradiction, since the diameter of P is 1. So, regular (d, t)-polygons, for $t = 8$, do not exist.

4. THE MAXIMUM VALUE OF THE PERIMETER OF THE CONVEX POLYGONS

For every convex t-polygon P of diameter 1, from [4] we have

$$L(P) \leqslant 2t\cdot\sin\left(\frac{\pi}{2t}\right)$$

with equality if, and only if P is a regular (1, t)-polygon. Hence, it holds

$$\ell(t) \leq 2t\cdot\sin\left(\frac{\pi}{2\tau}\right)$$

for every $t \geq 3$.

But, in the case that t has an odd divisor greater than 1, the existence of regular (1, t)-polygons has been shown. So, we have

$$\ell(t) = 2t\cdot\sin\left(\frac{\pi}{2t}\right)$$

for every $t = m2^n$, where m is an odd greater than 1 and $n = 0, 1.,$

Also, from the Proposition 3.2., we have the strict inequality

$$L(P) < 16\cdot\sin\left(\frac{\pi}{16}\right)$$

for every convex 8-polygon P of diameteer 1. Therefore, even if

$$\ell(8) = 16 = \sin\left(\frac{\pi}{16}\right)$$

this value is not attained by any convex 8-polygon of diameter 1.

In order to find $\ell(4)$ we prove the following:

4.1. Proposition. *Among the convex 4-polygons of diameter 1, the maximum perimeter has only the* $(3 \mid 1, 1, 2)$ *4-polygon.*

Proof. Let A, B, C and D be the vertices of a convex quadrilateral P of diameter 1 and maximal perimeter. Let also AC be a diagonal of P of length 1.

We put

$$2\alpha_1 = (AB) + (BC)$$

and

$$2\alpha_2 = (AD) + (DC)$$

Assuming that α_1 is constant, the vertex B is a point of the ellipse with foci the points A, C and sum $2\alpha_1$. Similarly, D is a point of the ellipse with foci the points A, C and sum $2\alpha_2$. Let AB be the side of P with the greatest length. From the maximality of $\alpha_1 + \alpha_2$, applying elementary geometrical methods, if follows that BD is vertical to AC and

$$(AB) = (BD) = 1$$

Let now ϕ be the angle CAB, We may suppose that $\phi \geqslant \frac{\pi}{4}$, for otherwise we set ϕ to be the angle DBA. Since the diameter of P is 1, we have $(BC) \leqslant 1$ and this implies that $\phi \leqslant \frac{\pi}{3}$. Hence,

$$\frac{\pi}{4} \leqslant \phi \leqslant \frac{\pi}{3}.$$

Then, we get

$$L(P) = (AB) + (BC) + (CD) + (DA) =$$

$$= 1 + 2\sin\left(\frac{\phi}{2}\right) + \sqrt{3 - 2\sin\phi - 2\cos\phi} + 2\sin\left(\frac{\pi}{4} - \frac{\phi}{2}\right)$$

and so

$$\frac{d}{d\phi} L(P) = \cos\left(\frac{\phi}{2}\right) + \frac{\sin\phi - \cos\phi}{\sqrt{3 - 2\sin\phi - 2\cos\phi}} - \cos\left(\frac{\pi}{4} - \frac{\phi}{2}\right).$$

Now, since $\phi \geq \frac{\pi}{4}$ the strict inequality

$$\frac{d}{d\phi} L(P) > 0$$

is equivalent to the inequality

$$\sqrt{2}\cos\left(\phi - \frac{\pi}{4}\right) + \frac{3\sqrt{2} - 2}{4} > 0$$

which is true for every $\phi \in \left[\frac{\pi}{4}, \frac{\pi}{3}\right]$.

Hence, $L(P)$ is a strictly increasing function of ϕ in $\left[\frac{\pi}{4}, \frac{\pi}{3}\right]$, which implies that for every 4-polygon P it holds

$$L(P) \leqslant 2 + 4\sin\left(\frac{\pi}{12}\right) = 2 + 2\sqrt{2 - \sqrt{3}}$$

with equality if, and only if, P is a $(3 \mid 1, 1, 2)$ 4-polygon.

4.2. Corollary. *$\ell(4) = 2 + 2\sqrt{2 - \sqrt{3}}$,*

and this value is realised only by the $(3 \mid 1, 1, 2)$ 4-polygon.

Proof. This is an immediate consequence of the Proposition 4.1.

Now we prove a result concerning the $(1, m, 2^n)$-polygons, according to the definition 2.1.

4.3. Theorem. *For $n \geq 3$, among the $(1, m, 2^n)$-polygons, the maximum perimeter has only the $(3 \mid k, k, k+1)$ 2^n-polygons, the case that $2^n = 3k + 1$, or the $(3 \mid k, k+1, k+1)$ 2^n-polygon in the case that $2^n = 3k + 2$.*

First we prove some Lemmas.

4.3.1. Lemma. *Let $f(x)$ be a strictly concave real function. If α, β, γ and δ are in the domain of the function $f(x)$, such that*

$$\alpha + \beta = \gamma + \delta$$

and $\alpha < \gamma \leqslant \delta < \beta$, *then*

$$f(\alpha) + f(\beta) < f(\gamma) + f(\delta).$$

Proof. Since $f(x)$ is a strictly concave function it follows that $f'(x)$ is strictly decreasing function and hence, from the mean value theorem, we have

$$\frac{f(\gamma) - f(\alpha)}{\gamma - \alpha} > \frac{f(\beta) - f(\delta)}{\beta - \delta}.$$

But $\gamma - \alpha = \beta - \delta > 0$, and so we get the result.

4.3.2. Lemma. *Let* t *and* m *be two given integers, such that* $3 \leqslant m \leqslant t$, *and* m *odd. If* P *is an* $(1, m, t)$*-polygon of maximum perimeter, then,* P *is an* $(m \mid k_1, k_2, \ldots, k_m)$ *t-polygon, with*

$$|k_i - k_j| \leqslant 1,$$

for every $i, j = 1, 2, \ldots, m$.

Proof. Let P^* denote the union of the 1-arcs consttructed on the sides of P, exterior to P. P is an $(1, m, t)$-polygon, and so P^* consists of m the number equal arcs each of length $\frac{\pi}{m}$.

Let $A_0, A_1, \ldots, A_r$ be the successive vertices of P which belong to one of these arcs and ϕ_i be the length of the arc $A_{i-1}A_i$, $i = 1, \ldots, r$.

The function $f(x) = \sin x$ is strictly concave in $\left(0, \frac{\pi}{2}\right)$ and since

$$\phi_1 + \phi_2 + \ldots + \phi_r = \frac{\pi}{m},$$

we have that

$$\sum_{i=0}^{r-1} (A_iA_{i+1}) = 2 \sum_{i=1}^{r} \sin\left(\frac{\phi_i}{2}\right) \leqslant 2r \cdot \sin\left(\frac{\pi}{2rm}\right),$$

with equality if, and only if,

$$\phi_1 = \phi_2 = \ldots = \phi_r = \frac{\pi}{rm}.$$

This implies that each of the m arcs of P^* is divided into equal parts for otherwise, P would not be of maximal perimeter. Thus, P is an $(m \mid k_1, k_2, \ldots, k_m)$ t-polygon, with perimeter

$$L(P) = \sum_{i=1}^{m} 2k_i \cdot \sin\left(\frac{\pi}{2mk_i}\right) \tag{1}$$

Now, the function $f(x) = x\sin\dfrac{\pi}{2mx}$ is strictly concave in $[1, +\infty)$ because

$$f''(x) = -\frac{1}{x^3}\left(\frac{\pi}{2m}\right)^2 \cdot \sin\left(\frac{\pi}{2mx}\right) < 0,$$

for every $x \geq 1$.

If we assume that for two positive integers it holds

$$k_i - k_j \geqslant 2,$$

we set $k_i' = k_i - 1$ and $k_j' = k_j + 1$. So, applying the Lemma 4.3.1. we get

$$k_i\sin\left(\frac{\pi}{2mk_i}\right) + k_j\sin\left(\frac{\pi}{2mk_j}\right) < k_i'\sin\left(\frac{\pi}{2mk_i'}\right) + k_j'\sin\left(\frac{\pi}{2mk_j'}\right) \quad (2)$$

The relations (1) and (2) contradict the maximality of $L(P)$, hence, for every $i, j = 1, 2, \ldots, m$ we have

$$|k_i - k_j| \leqslant 1$$

4.3.3. Lemma. *Let* t, m *and* P *be as in the Lemma 4.3.2. If* m *is not a divisor of* t*, then,* P *is an* $(m \mid k, k, \ldots, k, k+1, k+1, \ldots, k+1)$ *t-polygon.*

Proof. From the Lemma 4.3.2, we have that P is an $(m \mid k_1, k_2, \ldots, k_m)$ t-polygon, with

$$|k_i - k_j| \leqslant 1$$

for every $i, j = 1, 2, \ldots, m$.

The integers k_i, $i = 1, 2, \ldots, m$, are not all equal, for otherwise m would be a divisor of t, Hence, there exist $i, j \in \{1, 2, \ldots, m\}$ such that $k_j < k_j$. We put $k_i = k$, then $k_j = k + 1$.

Thus, for any $r = 1, 2, \ldots, m$, with $r \neq i,j$, $k_r = k$ or $k_r = k + 1$ and this implies the result.

4.3.4. Remark. Given the positive integers t and m as in the previous Lemma, there exist unique positive integers k and υ, such that

$$t = km + \upsilon, \quad 0 < \upsilon < m$$

If $1 \leqslant \upsilon \leqslant m - 2$, then

$$mk \leqslant mk + \upsilon - 1 < mk + \upsilon + 2 \leqslant mk + m$$

The function $f(x)$ is strictly concave, so, we have the relations

$$\frac{m - \upsilon}{m} \cdot f(mk) + \frac{\upsilon}{m} \cdot f(mk + m) \leqslant \frac{2}{\upsilon + 2} \cdot f(mk) + \frac{\upsilon}{\upsilon + 2} \cdot f(mk + \upsilon + 2) \quad (3)$$

$$\frac{2}{\upsilon + 2} \cdot f(mk) + \frac{\upsilon}{\upsilon + 2} \cdot f(mk + \upsilon + 2) \leqslant \frac{2}{3} \cdot f(mk + \upsilon - 1) + \frac{1}{3} \cdot f(mk + \upsilon + 2) \quad (4)$$

In (3) the equality holds exactly when $\upsilon = m - 2$ and in (4) exactly when $\upsilon = 1$, hence, the inequality (2) is true.

If $\upsilon = m - 1$, then, we put $c = 2^n + 1 = mk + m$, and the relation (2) becomes

$$\frac{1}{m} \cdot f(c - m) + \frac{m - 1}{m} \cdot f(c) < \frac{2}{3} \cdot f(c - 2) + \frac{1}{3} \cdot f(c + 1) \quad (5)$$

Next, we consider the function

$$g(m) = \frac{1}{m} \cdot f(c - m) + \frac{m - 1}{m} \cdot f(c) \quad (6)$$

and we have

$$g'(m) = \frac{1}{m} \left[\frac{f(c) - f(c - m)}{m} - f'(c - m) \right].$$

Since the function $f'(x)$ is strictly decreasing, by the mean value theorem, it follows that $g'(m) < 0$, and so the function $g(m)$ is strictly decreasing. Thus, for every $m \geq 5$, we have

$$g(m) < g(3) = \frac{1}{3} \cdot f(c - 3) + \frac{2}{3} \cdot f(c) \quad (7)$$

Now, the relation

$$\frac{1}{3} f(c - 3) + \frac{2}{3} f(c) < \frac{2}{3} f(c - 2) + \frac{1}{3} f(c + 1) \quad (8)$$

is equivalent to the inequality

$$\frac{f(c) - f(c - 2)}{2} < \frac{f(c + 1) - f(c - 3)}{4},$$

which is true since the function $f(x)$ is strictly increasing and strictly concave (see for example [1] page 47).

Hence, from the relations (6), (7) and (8), it follows that the realtion (5) is true. But this implies that the relation (2) is true and so P is a $(3 \mid k, k, k+1)$ 2^n-polygon (in the case that $2^n \equiv 1 \pmod 3$).

CASE II : n is odd. In this case $2^n = 3p + 2$, for some integer $p \geqslant 2$. As in the case I, it suffices to show that

$$\frac{m-\upsilon}{m} f(mk) + \frac{\upsilon}{m} f(mk+m) < \frac{2}{3} f(3p+3) + \frac{1}{3} f(3p)$$

for $m \geqslant 5$. But $3p + 2 = mk + \upsilon$, so the above is equivalent to

$$\frac{m-\upsilon}{m} f(mk) + \frac{\upsilon}{m} f(mk+m) < \frac{2}{3} f(mk+\upsilon+1) + \frac{1}{3} f(mk+\upsilon-2) \quad (9)$$

If $2 \leqslant \upsilon \leqslant m-1$, then,

$$mk \leqslant mk + \upsilon - 2 < mk + \upsilon + 1 \leqslant mk + m$$

and since the function $f(x)$ is strictly concave, we have the relations

$$\frac{m-\upsilon}{m} f(mk) + \frac{\upsilon}{m} f(mk+m) \leqslant \frac{\upsilon}{\upsilon+1} f(mk+\upsilon+1) + \frac{1}{\upsilon+1} f(mk) \quad (10)$$

$$\frac{\upsilon}{\upsilon+1} f(mk+\upsilon+1) + \frac{1}{\upsilon+1} f(mk) \leqslant \frac{2}{3} f(mk+\upsilon+1) + \frac{1}{3} f(mk+\upsilon-2) \quad (11)$$

The equality holds in (10) exactly when $\upsilon = m - 1$, and in (11) exactly $\upsilon = 2$. Hence, the strict inequality (9) is true.

If $\upsilon = 1$, then, we put $c = 2^n - 1 = mk$, and the relation (9) becomes

$$\frac{m-1}{m} f(c) + \frac{1}{m} f(c+m) < \frac{2}{3} f(c+2) + \frac{1}{3} f(c-1) \quad (12)$$

Now, we consider the function

$$g(m) = \frac{m-1}{m} f(c) + \frac{1}{m} f(c+m) \quad (13)$$

which has first derivative

$$g'(m) = -\frac{1}{m}\left[\frac{f(c+m)-f(c)}{m} - f'(c+m)\right] < 0$$

as in the case I. Thus, the function $g(m)$ is strictly decreasing and since $m \geqslant 5$, we have

$$g(m) < g(3) = \frac{2}{3} f(c) + \frac{1}{3} f(c+3) \tag{14}$$

Finally, since the function $f(x)$ is strictly increasing and strictly concave, it holds

$$\frac{f(c+3) - f(c-1)}{4} < \frac{f(c+2) - f(c)}{2} \tag{15}.$$

The relations (13), (14) and (15) prove the ineguality (12), and so the relation (9) is true.

4.4. Proposition. *The regular 2^n-polygon of diameter 1 has perimeter strictly less than the perimeter of the $(3 \mid k, k, k+1)$ 2^n-polygon in the case that $2^n = 3k+1$, or the perimeter of the $(3 \mid k+1, k+1, k)$ 2^n-polygon in the case that $2^n = 3k+2$.*

Proof. The perimeter of the regular 2^n-polygon of diameter 1 is equal to

$$2^n \sin\left(\frac{\pi}{2^n}\right),$$

We consider the function

$$f(x) = x \sin\left(\frac{\pi}{2x}\right)$$

then, it is sufficient to show that

$$f\left(\frac{3k+1}{2}\right) < \frac{2}{3} f(3k) + \frac{1}{3} f(3k+3),$$

in the case that $2^n = 3k+1$, and

$$f\left(\frac{3k+2}{2}\right) < \frac{2}{3} f(3k+3) + \frac{1}{3} f(3k),$$

in the case that $2^n = 3k+2$. But, these relations are obvious since the function $f(x)$ is strictly increasing (see the proof of the theorem 4.4) and $k \geqslant 2$.

4.5. Remark. Considering the $(3 \mid k, k, k+1)$ or the $(3 \mid k+1, k+1, k)$ 2^n-polygons, we conclude that

$$\ell(2^n) \geqslant 4k\sin\left(\frac{\pi}{6k}\right) + 2(k+1)\sin\left(\frac{\pi}{6(\kappa+1)}\right)$$

in the case that $2^n = 3k+1$, and

$$\ell(2^n) \geqslant 2k\sin\left(\frac{\pi}{6k}\right) + 4(k+1)\sin\left(\frac{\pi}{6(\kappa+1)}\right)$$

in the case that $2^n = 3k+2$.

Now, we quote the following:

Problem: *"Does there exist any convex 2^n-polygon of diameter 1 which have perimeter strictly greater than the perimeter of the $(3 \mid k, k, k+1)$ or the $(3 \mid k+1, k+1, k)$ 2^n-polygon, for $n \geqslant 3$?". (For $n = 2$ does not exist any).*

According to the theorem 4.3. the possible existence of such a polygon must be searched outside the class of the $(1, m, 2^n)$-polygons.

An immediate consequence of the remark 4.5 and the inequality

$$\ell(2^n) \leqslant 2t \cdot \sin\left(\frac{\pi}{2t}\right)$$

for $t \geqslant 3$, is the following:

4.6. Theorem. $\ell(2^n) = \pi - \dfrac{\pi^3}{24 \cdot 2^{2n}} + 0(2^{-3n}).$

5. THE LARGEST PERIMETER OF THE CONVEX POLYGONS WITH EQUAL SIDES

A well known problem (see for example [3]) is to find the number $P_2(t)$ defined as the least positive number such that any convex domain of diameter 1, can have its boundary divided into t sets, each of diameter at most $P_2(t)$. A closely related problem is to find the largest possible perimeter of a convex t-polygon of given diameter. Furthermore, we may restrict this problem in the class of the convex t-polygons with equal sides. In the case that t has an odd divisor strictly greater than 1, the answer is the square of diameter 1. However, if $t = 2^n$, with $n \geqslant 3$, the problem remains open.

Here, we make the following conjecture:

"Among the convex 2^n-polygons of diameter 1 and equal sides, only the regular polygons realise the maximum perimeter".

6. THE POLYGONS OF A GIVEN DIAMETER WITH THE GREATEST AREA

First we prove the following:

6.1. Theorem. *Among the t-polygons of diameter d, where t is an odd, only the regular t-polygons have the maximum area.*

Proof. Let P be a t-polygon of diameter 1 and of maximum area. Certainly, P is a convex t-polygon.

Now, for a given perimeter the maxumum area is realised only by the regular t-polygons. Also, since t is an odd integer, from [4] among the convex t-polygons of diameter 1, only the regular (1, t)-polygons have the maximum perimeter. But a regular t-polygon of diameter 1, since t is odd, is also a regular (1, t)-polygon. Hence, the polygon P is of maximum area if, and only if, it is regular.

Next, for the cases $t = 4$ and $t = 6$, we whall need a Lemma the proof of which is obvious.

6.2. Lemma. *Let Q be a quadrilateral with diagonals of length α and β respectively. Then, Q has maximum area $\dfrac{\alpha\beta}{2}$ if, and only if Q has orthogonal diagonals.*

6.3. Corollary. Among the 4-polygons of diameter 1, the maximum area realise all the 4-polygons with orthogonal diagonals each of length 1.

6.4. Proposition. *Among the hexagons of diameter 1, only the regular hexagon has maximum area.*

Proof. Let A_i, $i = 1, 2, ..., 6$, be the successive vertices of the hexagon P of diameter 1.

Let x, y, z be the lengths of the line segments A_1A_3, A_3A_5 and A_5A_1 respectively. Let also E_x, E_y and E_z denote the areas of the quadrilarals $A_1A_2A_3A_5$, $A_4A_5A_1A_3$ and $A_5A_6A_1A_3$ respectively.

From the Lemma 6.2 we have the relations

$$E_x \leqslant \frac{x}{2}, \quad E_y \leqslant \frac{y}{2}, \quad E_z \leqslant \frac{z}{2} \tag{1}$$

If E is the area of P and S is the area of the triangle $A_1A_3A_5$ then, we have from (1)

$$E = E_x + E_y + E_z - S \leqslant \frac{x+y+z}{2} + 2S,$$

with equality if, and only if A_1A_4, A_2A_5, A_3A_6 are orthogonal to A_3A_5, A_1A_3, A_1A_5 respectively. But

$$S = \sqrt{\tau(\tau - x)(\tau - y)(\tau - z)}$$

where $\tau = \dfrac{x+y+z}{2}$. Hence, considering the function $f(x, y, z) = \dfrac{\tau}{2} - S$ and differentiating, we find that it gets its maximum value exactly when

$$x = y = z = \frac{\sqrt{3}}{2} \tag{2}$$

Thus,

$$E \leqslant \frac{3\sqrt{3}}{8}$$

with equality if, and only if, A_1A_4, A_2A_5, A_3A_6 are orthogonal to A_3A_5, A_1A_3, A_1A_5 respectively, and the relation (2) holds. But this implies that the hexagon P has the maximum area exactly when it is regular.

6.5 Remark. If P is a regular 4n-polygon of diameter 1, then there are ifninitely many 4n-polygons of diameter 1 and the area of P, since we may remove a diagonal for example without increasing the diameter of P and without changing the area.

So we make the following conjectures:

Conjecture 1. *Among the t-polygons of diameter 1, the regular t-polygons realise the maximum area.*

Conjecture 2. *Let n be an odd integer, n > 3, then among the 2n-polygons of diameter 1, only the regular 2n-polygons realise the maximum area.*

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