

ON THE CONJUGATE EQUATION OF A SINGULAR DIFFERENTIAL OPERATOR

BY

P. D. Siafarikas

ABSTRACT. Let $H_1(\Delta)$ be the Banach space of all functions $f(z) = \sum_{n=1}^{\infty} f_n z^{n-1}$ which are analytic in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and satisfy the condition $\sum_{n=1}^{\infty} |f_n| < +\infty$.

In this paper we prove that the cokernel of the operator corresponding to the singular differential equation $z^2 y'(z) + a(z)y(z) = b(z)$ in the complex plane, consists of function which belong to $H_1(\Delta)$. Also we prove that the solutions of the differential equation $w''(z) + q(z)w(z) = 0$ are elements of $H_1(\Delta)$ under the assumption that $q(z)$ belong to $H_2(\Delta)$, i.e., the Hilbert space of functions $h(z) = \sum_{n=1}^{\infty} h_n z^{n-1}$ which are analytic in Δ and satisfy the condition $\sum_{n=1}^{\infty} |h_n|^2 < \infty$. These results improves previously known results.

KEY WORDS AND PHRASES. Singular differential operator, conjugate equation, Hardy Lebesgue space, cokernel of operator 1980 Mathematics subject classification codes. 34A20, 30D55.

1. INTRODUCTION

We consider the singular differential equation

$$z^2 y'(z) + a(z)y(z) = b(z) \quad (1.1)$$

where $a(z)$ is analytic in some neighborhood of the closed unit disc

$$\bar{\Delta} = \{z \in \mathbb{C} : |z| \leq 1\}$$

and $b(z)$ belong to $H_2(\Delta)$.

L.M. Hall [1] using the assumptions that $a(z)$ and $b(z)$ are analytic on $\bar{\Delta}$ and continuous on $\bar{\Delta}$, while $y(z)$ is analytic on Δ and continuously differentiable once on $\bar{\Delta}$, proved that the cokernel of the operator in (1.1) consists of functions which belong to the space $H_2(\Delta)$.

In this paper we shall prove that the cokernel of the operator in (1.1) consists of functions which belong to the space $H_1(\Delta)$. Obviously if $y(z) \in H_1(\Delta)$ then $y(z) \in H_2(\Delta)$.

Also in this paper we shall prove that the solutions of the differential equation

$$w''(z) + q(z)w(z) = 0 \quad (1.2)$$

belong to the space $H_1(\Delta)$, under the assumption that $q(z)$ is an element of $H_2(\Delta)$.

In a recent paper [4], Ch. Pommerenke gave the following two conditions

$$(i) \quad \iint_{\Delta} (1 - |z|^2)^3 \cdot |q(z)|^2 \frac{1 - |\zeta|}{|1 - \bar{\zeta}z|^2} dx dy < \alpha,$$

where $\zeta \in \Delta$, and $\alpha > 0$, an absolute constant, or

$$(ii) \quad \frac{1}{\delta} \int_{1-\delta}^1 \int_{\theta-\delta}^{\theta+\delta} (1-r)^3 \cdot |q(re^{it})|^2 dt dr < \beta,$$

where $0 \leq \theta \leq 2\pi$, $0 < \delta < \delta_0$, and $\beta > 0$ an absolute constant, to prove that the solutions of Eq. (1.2) belong to the Hardy-Lebesgue space $H_2(\Delta)$.

The conditions given by Ch. Pommerenke are better than the condition given by us, but his result is weaker than our result.

The method used [3] is presented in the preliminaries below and reduces the study of Eq. (1.1), (1.2), in the Banach space $H_1(\Delta)$. This method was applied successfully in [3] for the study of the nonlinear equations of the form

$$(L + A)f(z) = G(z, f(z)), \quad (1.3)$$

where L is the differential operator

$$Lf(z) = z^d \cdot \frac{d^{(k)}f(z)}{dz^k} + \sum_{i=1}^k \phi_i(z) \frac{d^{k-i}f(z)}{dz^{k-i}}, \quad d < k, \quad d \in \mathbb{Z},$$

defined in the space $H_2(\Delta)$, A is a bounded operator on $H_2(\Delta)$ and $G(z, f(z))$ is an analytic function in $f(z)$.

2. PRELIMINARIES

Denote an abstract separable Hilbert space over the complex field by H , an orthonormal basis in H by $e_n\}_{n=1}^{\infty}$ and the unilateral shift operator on H by $V(V : Ve_n = e_{n+1})$. We can easily see that the following statements holds:

(i) Every value z in the open unit disc Δ , is an eigenvalue of $V^*(V^*e_n = e_{n-1}$ for $n > 1, V^*e_1 = 0)$ the adjoint of V . The eigenlements $f_z = \sum_{n=1}^{\infty} z^{n-1} e_n$ form a complete system in H , in the sence that

if f is orthogonal to f_z , for every $z : |z| < 1$ then $f = 0$.

(ii) The mapping $f(z) = (f_z, f), f \in H$, where (\cdot, \cdot) denote the scalar product in H , is an isomorphism from H onto $H_2(\Delta)$.

(iii) The diagonal operator $C_0 : C_0e_n = ne_n, n = 1, 2, \dots$, has a self-adjoint extension in H with a compact inverse

$$B : Be_n = \frac{1}{n} e_n, n = 1, 2, \dots,$$

(see proposition 2 in ref. 2). Moreover if $f(z) = (f_z, f)$ then:

$$z^n f(z) = (f_z \cdot V^n f) \tag{2.1}$$

$$f^{(n)}(z) = (f_z \cdot (C_0 V^*)^n f). \tag{2.2}$$

$$z^2 f'(z) = (f_z \cdot (V^2 C_0 V^*) f) \tag{2.3}$$

and

$$C_0 V - V C_0 = V \tag{2.4}$$

in the sence that $(C_0 V - V C_0) f = V f$ for every f in $D(C_0)$, the definition domain of C_0 .

(iv) If $a(z) = \sum_{n=1}^{\infty} a_n z^{n-1}$ is analytic in some neighborhood of $\bar{\Delta}$, then $a^*(V) = \sum_{n=1}^{\infty} a_n V^{n-1}$ is a bounded operator on H and by the representation (ii) we have $a(z)f(z) = (f_z \cdot a^*(V)f)$, where $a^*(V) = \sum_{n=1}^{\infty} \bar{a}_n V^{n-1}$ and \bar{a}_n the complex conjugate of a_n [2].

Now we consider the linear manifold of all elements $f(z) = \sum_{n=1}^{\infty} a_n z^{n-1}$ in $H_2(\Delta)$ which satisfy the condition $\sum_{n=1}^{\infty} |a_n| < \infty$. By defining the new norm $\|f(z)\|_{H_1(\Delta)} = \sum_{n=1}^{\infty} |a_n|$ this manifold becomes a well known Banach space $H_1(D)$. We denote by H_1 the corresponding by the isomorphism $f(z) = (f_z \cdot f)$ Banach space in H . Every element $f = \sum_{n=1}^{\infty} (f, e_n)e_n$ in H_1 satisfies the condition.

$$\|f\|_1 = \|f(z)\|_{H_1(\Delta)} = \sum_{n=1}^{\infty} |(f, e_n)|. \quad (2.5)$$

Every bounded operator on $H_2(\Delta)(H)$ is defined on $H_1(\Delta)(H_1)$ and maps in general elements of $H_1(\Delta)(H_1)$ into $H_2(\Delta)(H)$. We give below some properties which follow easily

(v) H_1 is invariant under the operator V, V^* and

$$\|V\|_1 = \|V^*\|_1 = 1 \quad (2.6)$$

where $\|A\|_1$ means the norm of an operator A on H_1 .

(vi) H_1 is invariant under the bounded diagonal operator

$$B e_n = \frac{1}{n} e_n, \quad n = 1, 2, \dots, \text{ on } H. \text{ Moreover } \|B\|_1 = \|B\| = 1 \quad (2.7)$$

(vii) For every element $f(z) = \sum_{n=1}^{\infty} \alpha_n z^{n-1}$ in $H_1(\Delta)$ the uniform limit of the sequence $\sum_{i=1}^n \bar{\alpha}_i V^{i-1}$ exists and defines a bounded operator

$$f^*(V) = \bar{a}_1 + \bar{a}_2 V + \bar{a}_3 V^2 + \dots \quad (2.8)$$

on H_1 Moreover

$$\|f^*(V)\|_1 = \|f\|_1 \quad (2.9)$$

(viii) The null space of V^* in H belongs to H_1

3. MAIN RESULTS

The main results are the following:

Theorem A Let $\alpha(z), b(z)$, be elements of the space $H_2(\Delta)$. Then, the cokernel of the operator corresponding to the singular differential equation

$$z^2 y'(z) + a(z) \cdot y(z) = b(z) \quad (3.1)$$

consists of functions which belong to the space $H_1(\Delta)$.

Proof. According to the preliminaries the Eq. (3.1) is equivalent to the following operator equation on H .

$$T y = b \quad (3.2)$$

where $T = V^2 C_0 V^* + a^*(V) = \sum_{n=1}^{\infty} \bar{\alpha}_n V^{n-1}$ and $y \in H, b \in H$.

Let $f \in \ker T^*$, ie $T^* f = 0$

$$[VC_0V^{*2} + a(V^*)] f = 0$$

where $a(V^*)$ is the adjoint of $a^*(V)$ and $f \in H$.

Since $VC_0V^{*2} = C_0V^* - V^*$ and C_0 is invertible from E_q (3.3) we have

$$[V^* + Ba_1(V^*)]f = 0, \quad (3.4)$$

where $a_1(V^*) = a(V^*) - V^*$.

From Wq. (3.4) it follows that

$$VV^*f = -VBa_1(V^*)f \quad (3.5)$$

Since the operator VV^* is a projection operator which projects on $HO\{e_1\}$, from Eq. (3.5) we have

$$f = (f, e_1)e_1 - VBa_1(V^*)f. \quad (3.6)$$

Therefore, from Eq. (3.6) since $(f, e_1)e_1 \in H_1$ and $-VBa_1(V^*)f \in H_1$ follows that $f \in H_1$.

Finally, according to the isomorphism $f(z) = (f_2, f)$, $f(z)$ belongs to $H_1(\Delta)$.

Theorem B. *If $q(z)$ belong to $H_2(\Delta)$, then every solution of the equation*

$$w''(z) + q(z) \cdot w(z) = 0 \quad (3.7)$$

belong to $H_1(\Delta)$.

Proof. According to the preliminaries Eq. (3.7) is equivalent to the following operator equation on H

$$[C_0V^{*2} + q^*(V)] w = 0 \quad (3.8)$$

where $q^*(V) = \bar{q}_0 + \bar{q}_1V + \bar{q}_2V^2 + \dots$, and $w \in H$.

After some manipulation Eq. (3.8) can be written:

$$(I + VK)w = c \cdot e^2 \quad (3.9)$$

where $K = VB_1q^*(V)$.

It is not difficult to prove that the operator K in Eq. (3.9) leaves invariant the space H_1 , that the operator VK is compact operator

and the zero is not an eigenvalue of $(I + VK)$ in H_1 . Therefore, according to the Fredholm alternative, we obtain from Eq. (3.9) that

$$w = c \cdot (I - VK)^{-1} e_2 \quad (3.10)$$

Since e_2 belong to H_1 it follows from the above that w in Eq. (3.10) belong also to H_1 . Therefore, according to the isomorphism $w(z) = (w_z, w)$, $w(z)$ belong to $H_1(\Delta)$.

Remark 3.1. Theorem B can also be proved easily as follows:

If $q(z) = \sum_{n=0}^{\infty} \alpha_n z^n$, $w(z) = \sum_{n=0}^{\infty} b_n z^n$ then the differential equation (3.7) implies

$$(n+1)(n+2)b_{n+2} = - \sum_{k=0}^n \alpha_{n-k} b_k$$

From this we get

$$|b_{n+2}| \leq \frac{1}{(n+1)^2} \cdot \sum_{k=0}^n |\alpha_{n-k}| \cdot |b_k| \quad (3.11)$$

The inequality (3.11) yields

$$|b_{n+2}|^2 < \frac{1}{(n+1)^4} \cdot \left(\sum_{k=0}^n |\alpha_k|^2 \right) \cdot \sum_{k=0}^n b_k^2, \quad n = 0, 1, 2, \dots$$

This immediately gives $|b_n| = O(1)$, whence

$$|b_{n+2}| < \frac{\text{const}}{n^2} \cdot \left(\sum_{k=0}^{\infty} a_k^2 \right)^{1/2}.$$

Theorem B also holds if the sequence $\left\{ \frac{a_n}{n^2} \right\}$ belong to $l_1(1, \infty)$, i.e., the Banach space of sequences $\{f_n\}_{n=1}^{\infty}$ which satisfy the condition $\sum_{n=1}^{\infty} |f_n| < \infty$.

ACKNOWLEDGMENT

The author wants to express his appreciation to Professor E.K. Ifantis for most fruitful discussions and many helpful suggestions.

REFERENCES

1. L.M. Hall, A, characterization of the Cokernel of a singulara Fredholm Differential operator, J. Differential Equations, 24, (1977), 1-7.

2. E.K. Ifantis, An Existence Theory for Functional-Differential Equations and Functional Differential systems, *J. Differential Equations*, 29 (1978), 86-104.
3. E.K. Ifantis, Analytic; colutions for nonlinear differential equations, (submeten for publication).
4. Ch. Pommerenke, On the mean growth of the solutions of complex linear differential equations in the disk, *Complex Variables Theory Appl.* 1 (1982/83), 23-38.

(Received by the editors, November 6, 1985)

Department of Mathematics, University of Patras, Patra, Greece.