Bulletin LMS 29 (1997), 4, 415-424

Infinite patterns that can be avoided by measure

MIHAIL N. KOLOUNTZAKIS[1][2]
School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, USA
E-mail: koloun t@math.uiuc.edu

June 1996

Abstract

A set A of real numbers is called universal (in measure) if every measurable set of positive measure necessarily contains an affine copy of A. All finite sets are universal but no infinite universal sets are known. Here we prove some results related to a conjecture of Erdős that there is no infinite universal set. For every infinite set A there is a set E of positive measure such that $(x + tA) \subseteq E$ fails for almost all (Lebesgue) pairs (x, t). Also the exceptional set of pairs (x, t) (for which $(x + tA) \subseteq E$) can be taken to project to a null set on the t-axis. Last, if the set A contains large subsets whose minimum gap is large (in a scale-invariant way) then there is $E \subseteq \mathbf{R}$ of positive measure which contains no affine copy of A.

MSC (1991): 28A12

0. Introduction

0.1 Universal sets. Let us call a set A of real numbers *universal (in measure)* if every measurable $E \subseteq \mathbf{R}$ with positive Lebesgue measure necessarily contains an affine copy of A. That is, for every $E \subseteq \mathbf{R}$ with $\mu(E) > 0$ there are $x, t \in \mathbf{R}$ such that

$$x + tA := \{x + ta : \ a \in A\} \subseteq E.$$

By looking near a point of high density of E it is easy to see that all finite sets are universal.

[1]Supported by NSF grant DMS-9304580
[2]Current address: Department of Mathematics, 1409 W. Green St., University of Illinois, Urbana, IL 61801

Erdős [3] has asked whether there exist any infinite universal sets but none is known. Useful references can be found in [2, p. 184].

0.2 It is worth mentioning that the key lemma in a recent paper of Lebedev and Olevskiĭ [6] is of a similar nature, dealing with the universality of finite sets. For any set $E \subseteq \mathbf{R}$ with $\mu(\partial E) > 0$, and for every two disjoint finite sets $B = \{b_1, \ldots, b_m\}$ and $W = \{w_1, \ldots, w_n\}$, there are $x, t \in \mathbf{R}$ such that $(x+tB) \subseteq E$ and $(x+tW) \subseteq E^c$. Here ∂E is the essential boundary of E, that is all points in any neighborhood of which there is positive measure of both E and E^c (the complement of E). This is used to prove that all bounded operators $L^p(\mathbf{R}^d) \to L^p(\mathbf{R}^d)$, for $p \neq 2$, of the type $f \to (\mathbf{1}_E \cdot \widehat{f})^\vee$ (the idempotent multipliers), necessarily have E equal to an open set up to measure 0. Here $\mathbf{1}_E$ is the indicator function of the set E, and $\widehat{f}$ and $f^\vee$ are the Fourier and inverse Fourier transform of f respectively.

0.3 Known classes of non-universal sets. Though no infinite universal set is known, we know of some classes of infinite sets which are not universal.

Falconer [4] has proved that any sequence $\{x_n\}$ of positive reals decreasing to 0 is not universal if

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1. \qquad (1)$$

Falconer constructs a Cantor-type set with positive measure which avoids all affine copies of a sequence $\{x_n\}$ satisfying (1). It is unknown whether geometrically decreasing sequences such as $\{2^{-n}\}$ are universal or not.

Bourgain [1] shows that any set (in 3 dimensions) of the type $S \times S \times S$, where $S \subseteq \mathbf{R}$ is infinite, is not universal, defining universality in 3 dimensions to even allow affine copies scaled differently along the three axes. His method also gives that any set of reals of the type $S_1 + S_2 + S_3$, where the S_j are infinite, is not universal. He points out that a variant of his method also permits certain double sums like

$$\{2^{-n}\} + \{2^{-n}\} \qquad (2)$$

to be shown non-universal. Bourgain uses a probabilistic construction.

Komjáth [5] proves that for every infinite set $A \subseteq [0,1]$ there is another subset of $[0,1]$, of measure arbitrarily close to 1, that does not contain any translate of A (no scaling allowed).

We should say at this point that it is not even known whether all uncountable sets are not universal (see [2]). And, to the best of the author's knowledge, no non-universal sequence that decreases geometrically or faster is known. To be more precise, no sequence $x_n \downarrow 0$ is known, which satisfies $x_{n+1} \le \rho x_n$, for some fixed $\rho < 1$, and is provably non-universal.

0.4 New results. We show that for every infinite set A there is always another set E of positive measure which avoids "almost all" affine copies $x + tA$ of A (Theorems

1 and 2). We also prove non-universality in certain cases where the set contains structures with large minimum gaps (Theorem 3).

In §1 we prove that the exceptional set of pairs (x, t), that is the set of such pairs for which $(x + tA) \subseteq E$, has two-dimensional Lebesgue measure 0, for some appropriately constructed set E. Komjáth's result mentioned in §0.3 follows as an easy corollary.

In §2 we strengthen the result of §1 and prove that the projection of the exceptional set on the t-axis can be taken to have (one-dimensional) measure 0. This of course implies the result of §1 but the proofs of the two results are quite different and we present both of them.[3]

In §3 we prove that if the set A contains large structures with large minimum gap (relative to their scale) then A is not universal. From Theorem 3 we shall be able to give another proof of Falconer's result (1), as well as prove that sets of the type

$$\{2^{-n^\alpha}\} + \{2^{-n^\alpha}\}, \quad \text{with } 0 < \alpha < 2, \tag{3}$$

are not universal (compare with (2)). Falconer's criterion (1) cannot be used to handle the set (3) for $\alpha \geq 1$.

0.5 Notation. The letter C will be used to denote an absolute positive constant, not necessarily the same in all its occurrences, even in the same equation. We shall use $A + B$ to denote the set $\{a + b : \ a \in A, b \in B\}$ and tA for $\{ta : a \in A\}$. The expression $\mathbf{1}\,(\textit{condition})$ is equal to 1 if the condition is true and 0 otherwise. The fractional part of a real number x will, as usual, be denoted by $\{x\}$. We write $a \sim b$ for $\lim(a/b) = 1$ as an implicit parameter tends to a limit.

1. Almost no affine copy

1.1 Here we prove that almost all copies of a given set can be avoided by some set of positive measure.

Theorem 1 *Let $A \subseteq \mathbf{R}$ be infinite. Then there is a set $E \subseteq [0, 1]$, of measure arbitrarily close to 1, such that the set of pairs*

$$\{(x, t) : \ (x + tA) \subseteq E\}$$

[3] Theorem 2 has also been proved by Prof. M. Akcoglu (unpublished) with essentially the same proof.

has measure 0 *(Lebesgue measure in* $\mathbf{R}^2$*).*

Remark. It is sufficient to restrict the scaling parameter t in an interval $[\alpha, \beta]$, with $0 < \alpha < \beta < \infty$, because then we can write the whole scaling interval $(0, \infty)$ as a countable union of intervals of the above type and intersect the resulting sets. As the measure of those sets can be taken arbitrarily close to 1, so can the measure of their intersection.

1.2 Proof. It is obviously sufficient to prove the Theorem for the case of $A = \{a_1 > a_2 > \cdots\}$ being a sequence of positive reals decreasing to 0.

We fix an interval $[\alpha, \beta]$, $0 < \alpha < \beta < \infty$, of the scaling parameter and construct a random Cantor-type set $E \subseteq [0, 1]$ which will, with positive probability, have measure as close to 1 as we please and will contain almost no affine copy of the set A of the type $x + tA$, with x arbitrary and $t \in [\alpha, \beta]$.

The set E will be an intersection

$$E = \bigcap_{j=1}^{\infty} F_j,$$

where the indicator function of the random set of the j-th stage is given by

$$\mathbf{1}_{F_j} = \sum_{k=1}^{m_j} \epsilon_{j,k}\ \mathbf{1}_{I_{j,k}}, \quad \text{where} \quad I_{j,k} = \left[\frac{k-1}{m_j}, \frac{k}{m_j}\right).$$

The $\epsilon_{j,k} \in \{0, 1\}$ are a collection of independent indicator random variables with

$$\mathbf{Pr}\left[\epsilon_{j,k} = 1\right] = p_j.$$

(To state it otherwise, at the j-th stage we divide the unit interval into m_j equal subintervals and we keep each of them in F_j independently and with equal probability p_j.) The integers m_j are defined to be large enough so that $1/m_j$ is smaller than $1/2$ the minimum gap of the numbers $\alpha a_1, \ldots, \alpha a_j$. The probabilities p_j are taken such that

$$\prod_{j=1}^{\infty} p_j = q > 0, \quad \text{and} \quad \prod_{j=1}^{\infty} p_j^j = 0, \tag{4}$$

something which is clearly possible for every $q \in [0, 1)$.

Fixing $x \in [0, 1]$ we observe that $\mathbf{Pr}\left[x \in E\right] = q$, since x belongs to exactly one of the intervals $I_{j,k}$ for every value of j. Thus $\mathbf{E}\mu(E) = q$.

Now fix both $x \in [0, 1]$ and $t \in [\alpha, \beta]$, such that $(x + tA) \subseteq [0, 1]$. The crucial observation is that the numbers

$$ta_1, ta_2, \ldots, ta_j, \tag{5}$$

belong to different intervals of the collection $\mathcal{I}_j = \{I_{j,k} : \ k = 1, \ldots, m_j\}$, since the length of the $I_{j,k}$ has been chosen so small. For the set $x + tA$ to be contained in E it is necessary that the points in (5) all "survive" the j-th stage. In other words, all intervals of stage j which contain one of the points in (5) must be kept (their $\epsilon_{j,k}$ equal to 1). But, since there are exactly j of those intervals, the probability of this happening is exactly p_j^j, and the probability that this happens for all stages is equal to $\prod_{j=1}^{\infty} p_j^j = 0$.

Write $\varphi(x,t) = \mathbf{1}\left((x+tA) \subseteq E\right)$. We have proved that $\mathbf{E}\varphi(x,t) = 0$ for all pairs (x,t), with $t \in [\alpha, \beta]$, $x \in [0,1]$. Therefore

$$\begin{aligned} \mathbf{E}\mu\{(x,t) : \ (x+tA) \subseteq E\} &= \mathbf{E}\int_0^1 \int_\alpha^\beta \varphi(x,t)\ dtdx \\ &= \int_0^1 \int_\alpha^\beta \mathbf{E}\varphi(x,t)\ dtdx \ = \ 0, \end{aligned}$$

which proves that

$$\mu\{(x,t) : \ (x+tA) \subseteq E\} = 0, \text{ almost surely.} \tag{6}$$

Thus there exists a set E with $\mu(E) \geq q$ and satisfying (6) at the same time. Since $q \in [0,1)$ is arbitrary the theorem is proved. □

1.3 Komjáth's result. As a corollary of Theorem 1 we can give another proof of Komjáth's result [5] mentioned in §0.3. Indeed, let E be the set constructed in Theorem 1. By Fubini's theorem for almost all t

$$\mu\{x : \ (x+tA) \subseteq E\} = 0. \tag{7}$$

Fix such a t close to 1 and remove from E an open cover of the exceptional set in (7) with small measure. Call the resulting set E'. Now $\mu(E') > 0$ and E' contains no set of the form $x+tA$, $x \in \mathbf{R}$, since we have removed a neighborhood of an accumulation point of every such set. (We can of course assume that 0 is an accumulation point of A.) Scale E' by $1/t$ to obtain a set of positive measure which contains no translate of A. It is clear that the measure of the resulting set can be arbitrarily close to 1.

2. No translational copy at almost every scale

2.1 In this section we prove the following improvement of Theorem 1.

Theorem 2 *Let $A \subseteq \mathbf{R}$ be an infinite set. Then there exists $E \subseteq [0,1]$, with $\mu(E)$ aribitrarily close to 1, such that*

$$\mu\{t : \ \exists x \text{ such that } (x+tA) \subseteq E\} = 0. \tag{8}$$

The improvement over Theorem 1 is that we now know that the set of "bad" pairs (x,t), for which $(x+tA) \subseteq E$, projects to a null set on the t-axis.

The proof of Theorem 2 is rather different from that of Theorem 1. The set that we construct is a *deterministic* (and not random as was the case in §1) Cantor-type set. But we do use the random-like properties of a rapidly decreasing sequence of positive reals if looked at modulo a sufficiently small real number (the period T, in the proof of §2.3).

2.2 It should be pointed out that if we managed to have the set of bad pairs to project to a null set on the x-axis this would imply Erdős's conjecture that there are no infinite universal sets. To see this call $P \subseteq [0,1]$ the projection of the bad pairs on the x-axis and assume that $\mu(P) = 0$. Remove then from the set E an open cover of P of measure no larger than $\mu(E)/2$ to obtain a set of measure at least $\mu(E)/2$ that avoids *all* affine copies of the set A. (Again, we take here 0 to be an accumulation point of A.)

2.3 Proof. We may assume that $A = \{a_1, a_2, \ldots\}$, where a_j is a sequence of positive reals that decreases to 0. Fix an interval $[\alpha, \beta]$ of the scaling parameter t. Let the parameter $n \to \infty$. All we have to do is construct a set $E = E(n) \subseteq [0,1]$, of measure $\mu(E) = 1 - o(1)$, such that

$$\mu\{t \in [\alpha,\beta] :\ \exists x \text{ such that } (x+tA) \subseteq E\} \to 0, \tag{9}$$

as $n \to \infty$. Intersecting countably many such sets of measure sufficiently close to 1 one gets a set of positive measure that satisfies (8).

Slightly abusing notation, let us write

$$A_n = \{a_1, \ldots, a_n\}. \tag{10}$$

We may assume that the $a_j \in A_n$ decrease to 0 as rapidly as we please. That is we may assume that

$$\frac{a_{j+1}}{a_j} < \rho = \rho(n) < 1, \quad \text{for } j = 1, \ldots, n-1,$$

where the sequence $\rho(n) \to 0$ will be chosen later.

Define the set E to be periodic with period

$$T = T(n) = \rho\, \alpha\, a_n$$

and

$$E \cap [0,T] = (\epsilon T, T),$$

with $\epsilon = \epsilon(n) \to 0$ to be determined later. Clearly $\mu(E) \to 1$, as $n \to \infty$.

Call a scale t "bad" if the maximum gap of the numbers

$$ta_1 \bmod T, \ldots, ta_n \bmod T, \qquad (11)$$

(considered as points on a circle of length T) is greater than ϵT. We show that the measure of the set of bad scales is $o(1)$. This will prove the theorem since, if a certain t is not bad, then for all x at least one of the numbers

$$(x + ta_1) \bmod T, \ldots, (x + ta_n) \bmod T$$

falls in $[0, \epsilon T]$, which means that at least one of the numbers

$$x + ta_1, \ldots, x + ta_n$$

falls outside E and (9) follows.

Fix the numbers $\lambda_1, \mu_1, \ldots, \lambda_n, \mu_n \in [0,1]$ and consider the intervals $J_1 = [\lambda_1 T, \mu_1 T], \ldots, J_n = [\lambda_n T, \mu_n T] \subseteq [0, T]$. Define

$$\nu(J_1, \ldots, J_n) = \mu\{t \in [\alpha, \beta] : \ (ta_j \bmod T) \in J_j, \ \text{for } j = 1, \ldots, n\}.$$

It is clear that, for fixed $\lambda_1, \mu_1, \ldots, \lambda_n, \mu_n$ and for any given $\delta > 0$, we can take the ratio ρ so small so as to have

$$\left| \nu(J_1, \ldots, J_n) - (\beta - \alpha) \frac{|J_1|}{T} \cdots \frac{|J_n|}{T} \right| < \delta. \qquad (12)$$

In other words, for small ρ, the numbers in (11), behave approximately like uniformly distributed, independent random variables, when t moves from α to β, at least with respect to simple (i.e. finite unions of intervals) subsets of $[0, T]$. (See §2.4 for a proof of this.)

Let

$$k = \left\lceil \frac{10}{\epsilon} \right\rceil.$$

Define $I_i = \left[\frac{i}{k} T, \frac{i+1}{k} T\right]$, $i = 0, \ldots, k-1$, so that $|I_i| \sim \epsilon/10$, and consider the set

$$B = \{t \in [\alpha, \beta] : \ \exists i \in \{0, \ldots, k-1\} \ \forall j = 1, \ldots, n \ \ (ta_j \bmod T) \notin I_i\}.$$

Clearly every bad scale t belongs to B and thus it suffices to show that $\mu(B) \to 0$, as $n \to \infty$.

Let $X_1, \ldots, X_n$ be independent random variables uniformly distributed in $[0, T]$. As $\rho \to 0$ we have, because of (12),

$$\begin{aligned} \frac{1}{\beta - \alpha} \mu(B) &\to \mathbf{Pr}[\exists i \ \forall j \ \ X_j \notin I_i] \qquad (13) \\ &\le k \, \mathbf{Pr}[\forall j \ \ X_j \notin I_1] \\ &= k \, \mathbf{Pr}[X_1 \notin I_1]^n \\ &= k \left(1 - \frac{1}{k}\right)^n \\ &\le k \, e^{-n/k}. \end{aligned}$$

Now let $\epsilon = \epsilon(n) = 1/\sqrt{n}$, for example, so that $k \sim 10\sqrt{n}$ and

$$ke^{-n/k} \le C\sqrt{n}\, e^{-C\sqrt{n}} \to 0.$$

For each n we choose $\rho = \rho(n)$ to be so small (remember that the limit in (13) is taken for $\rho \to 0$) so as to have $\mu(B) \le C(\beta - \alpha)\sqrt{n}e^{-C\sqrt{n}}$, which tends to 0. □

2.4 Almost independence mod 1. Here we prove the claim made in (12). After scaling, it follows from the next lemma.

Lemma 1 *Let* $0 < \alpha < \beta$ *be fixed,* $0 < \rho < 1$, *and* $(\alpha\rho)^{-1} = x_N < x_{N-1} < \cdots < x_1$,

with

$$\frac{x_{i+1}}{x_i} \le \rho, \quad i = 1, \ldots, N-1.$$

Let also $I_1, \ldots, I_N$ *be subintervals of* $[0,1]$ *and* $\rho \to 0^+$ *with* $\rho = o\left(\frac{1}{N}\right)$. *Write*

$$S(I_1, \ldots, I_N) = \{t \in [\alpha, \beta] : \ \{tx_1\} \in I_1, \ldots, \{tx_N\} \in I_N\}.$$

Then

$$\mu S(I_1, \ldots, I_N) = (\beta - \alpha)|I_1| \cdots |I_N| + o(1).$$

Remark. In the application of the lemma in the proof of §2.3 N was kept fixed while $\rho \to 0^+$.
Proof. (The $O(\cdot)$- and $o(\cdot)$-implied constants may depend on α and β only.) Write

$$S_k = \{t \in [\alpha, \beta] : \ \{tx_k\} \in I_k, \ldots, \{tx_N\} \in I_N\}.$$

Clearly each S_k is a collection of subintervals of $[\alpha, \beta]$. We prove inductively, for $k = N, N-1, \ldots, 1$, that, apart from measure $O(\rho(N - k + 1))$, each S_k consists of a disjoint collection of equal-length intervals J_l^k, each of length $|I_k|/x_k$, of total length

$$(\beta - \alpha)|I_k| \cdots |I_N| + O(\rho(N - k + 1)).$$

This clearly proves the lemma.

For $k = N$ the set of $t \in [\alpha, \beta]$ such that $\{tx_N\} \in I_N$ consists of a collection of intervals J_l^N, of length $|I_N|/x_N$ each, plus at most two intervals of length at most $|I_N|/x_N \le \alpha\rho$ each. The number of the intervals J_l^N is $(\beta - \alpha)x_N + O(1)$, therefore the total length of the J_l^N is $(\beta - \alpha)|I_N| + O(\rho)$, as we had to prove.

Assume the assertion true for $k+1,\ldots,N$. Each of the J_l^{k+1} has length $|I_{k+1}|/x_{k+1}$. The set of $t \in J_l^{k+1}$ such that $\{tx_k\} \in I_k$ consists of a collection of intervals of length $|I_k|/x_k$ each plus at most two intervals of length at most $|I_k|/x_k$ each. Since the number of the J_l^{k+1} is $\lesssim (\beta-\alpha)x_{k+1}$ the total error committed at this stage is

$$\lesssim |I_k|(\beta-\alpha)\frac{x_{k+1}}{x_k} \le (\beta-\alpha)\rho$$

and the total length of the J_l^k is

$$(\beta-\alpha)|I_k|\cdots|I_N| + O(\rho(N-k+1)),$$

which concludes the proof. $\Box$

3. No affine copy for sets with large gaps

3.1 In this section we shall give a simple proof of the fact that a set of real numbers which contains a large subset with large gaps cannot be universal. Our result will imply Falconer's result (1) as well as the fact that a set of the type $B+B$, with $B = \{2^{-n^\alpha}\}$, for $0 < \alpha < 2$, cannot be universal. This is slightly stronger than the corresponding result (2) by Bourgain [1]. However our method does not give Bourgain's main result that a set of the type $A = S_1 + S_2 + S_3$, with S_j infinite, is not universal.

Theorem 3 *Let $A \subseteq \mathbf{R}$ be an infinite set which contains, for arbitrarily large n, a subset $\{a_1,\ldots,a_n\}$ with $a_1 > \cdots > a_n > 0$ and*

$$-\log\delta_n = o(n), \tag{14}$$

where

$$\delta_n = \min_{i=1,\ldots,n-1}\frac{a_i - a_{i+1}}{a_1}.$$

Then A is not universal.

3.2 Proof. Fix the scale interval $t \in [\alpha,\beta]$. Let n be as in the statement of the Theorem. For a given $E \subseteq [0,1]$ we write

$$\varphi(x) = \mathbf{1}\left(\exists t \in [\alpha,\beta] \text{ such that } (x + t\{a_1,\ldots,a_n\}) \subseteq E\right).$$

We shall construct a set $E = E(n) \subseteq [0,1]$ with $\mu(E) \to 1$ and $\int_0^1 \varphi(x)\,dx \to 0$ as $n \to \infty$. As in §2.3 this will suffice to prove the theorem: we intersect countably many such sets E with $\mu(E)$ very close to 1 so that the measure of the intersection is large, and with the sum of the quantities $\int_0^1 \varphi(x)\,dx$ being very small. We then remove from the intersection of the sets a small open cover of all those x's for which some $\varphi(x) = 1$ (see discussion in §2.2).

Let

$$k = \left\lceil \frac{10}{\alpha\, a_1\, \delta_n} \right\rceil .$$

Partition $[0,1]$ into k intervals of the type $I_i = \left[\frac{i}{k}, \frac{i+1}{k}\right)$, $i = 0,\ldots,k-1$, and put each I_i in E independently of the other intervals and with probability $p = p(n) \to 1^-$, which will be determined later in the proof. In other words

$$\mathbf{1}\,(x \in E) = \sum_{i=0}^{k-1} \epsilon_i \cdot \mathbf{1}\,(x \in I_i)\,,$$

where the $\epsilon_0,\ldots,\epsilon_{k-1} \in \{0,1\}$ are independent indicator random variables with $\mathbf{E}\epsilon_i = p$, $i = 0.\ldots,k-1$.

We have $\mathbf{E}\mu(E) = p$ and

$$\mathbf{E}\int_0^1 \varphi(x)\,dx = \int_0^1 \mathbf{Pr}\left[\exists t \in [\alpha,\beta] \text{ such that } (x + t\{a_1,\ldots,a_n\}) \subseteq E\right] dx. \tag{15}$$

We show that the probability in (15) tends to 0 uniformly in x, for $p \to 1^-$ properly chosen as a function of n, and under the assumption (14) made about δ_n.

Fix $x \in [0,1]$. To check whether there exists $t \in [\alpha,\beta]$ such that $(x+t\{a_1,\ldots,a_n\}) \subseteq E$ it is sufficient to check whether such a t exists in a *finite* set

$$S = S(x) = \{t_1,\ldots,t_N\} \subseteq [\alpha,\beta]. \tag{16}$$

This set consists exactly of those $t \in [\alpha,\beta]$ for which some $x + ta_j$, $j = 1,\ldots,n$, is in the set $\left\{0, \frac{1}{k}, \frac{2}{k}, \ldots, \frac{k-1}{k}, 1\right\}$. We have

$$N \le C\,\beta\,a_1\,n\,k. \tag{17}$$

This is so because each of the n points $t\,a_j$ traverses an interval of length at most $\beta\,a_1$, as the parameter t goes from α to β, and therefore meets at most $C\,\beta\,a_1\,k$ endpoints of the intervals I_i.

Since the length of the I_i has been chosen so small we have that for each $t \in [\alpha,\beta]$ the points $x + ta_j$, $j = 1,\ldots,n$, all belong to different intervals I_i, $i = 0,\ldots,k-1$. Exactly for this reason we have, for any fixed x and t,

$$\mathbf{Pr}\left[(x + t\{a_1,\ldots,a_n\}) \subseteq E\right] = p^n.$$

Thus, using the bound (17),

$$\begin{aligned}\mathbf{Pr}\,[\exists t \in S : (x + t\{a_1, \ldots, a_n\}) \subseteq E] &\le N\, p^n \\ &\le C\, \beta\, a_1\, n\, k\, p^n\end{aligned}$$

and therefore

$$\begin{aligned}\mathbf{E}\int_0^1 \varphi(x)\; dx &\le C\, \beta\, a_1\, n\, k\, p^n \\ &\le C\, \frac{\beta}{\alpha}\, \delta_n^{-1}\, n\, p^n.\end{aligned}$$

We want to have $p \to 1^-$ and at the same time $\delta_n^{-1} n p^n \to 0$, as $n \to \infty$.

Let $\ell = -\log p$. Then we want $\ell \to 0^+$ and

$$n\,\ell - \log n + \log \delta_n \to +\infty. \tag{18}$$

As we are free to take $\ell \to 0^+$ arbitrarily slowly it is necessary and sufficient for (18) to hold that we have $-\log \delta_n = o(n)$, which is the assumption of our Theorem.

As the random variables $\mu(E)$ and $\int_0^1 \varphi(x)dx$ are always in the range $[0, 1]$, and we have $\mathbf{E}\mu(E) \to 1$ while $\mathbf{E}\int_0^1 \varphi(x)dx \to 0$, we deduce that there exists a set E with $\mu(E) = 1 - o(1)$ and $\int_0^1 \varphi(x)dx = o(1)$, as we had to show. □

3.3 Falconer's result as a corollary of Theorem 3. Suppose $X = \{x_1, x_2, \ldots\}$ is a sequence of positive reals that decreases to 0 and is such that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$. For aribitrarily large n we shall extract $A = \{a_1, \ldots, a_n\} \subseteq X$ for which $-\log \delta_n = o(n)$, so that Theorem 3 shows that X is not universal. Let $\rho \to 1^-$ and for each ρ we construct a set $A \subseteq X$ with $|A| = n$ (n will tend to infinity as $\rho \to 1^-$).

Since $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$ there is $M \in \mathbf{N}$ such that for all $k \ge M$, $\frac{x_{k+1}}{x_k} > \rho$. Consider the intervals of the type $[\rho^{\nu+1}, \rho^\nu)$, for ν large enough to have $\rho^\nu < x_M$, say for $\nu \ge \nu_0$. Every such interval contains then at least one point of X. Call x'_ν any point of X that belongs to $[\rho^{\nu+1}, \rho^\nu)$ and define

$$a_j = x'_{2j+\nu_0},\ j = 1, 2, \ldots, n.$$

(That is, we put in A one point of X from every other interval $[\rho^{\nu+1}, \rho^\nu)$, for $\nu \ge \nu_0$.) We have

$$\delta_n \ge C \frac{(1-\rho)\,\rho^{2n+\nu_0+C}}{\rho^{\nu_0+C}} \ge C\,(1-\rho)\,\rho^{2n+C},$$

and

$$-\log \delta_n \le C - \log(1-\rho) + (-\log \rho)\cdot(2n + C),$$

which is $o(n)$ if $\rho \to 1^-$ is chosen so that $-\log(1-\rho) = o(n)$. □

3.4 Sets of the type $B+B$, with $B = \{2^{-n^\alpha}\}$ and $0 < \alpha < 2$. To prove using Theorem 3 that such a set is not universal we shall exhibit arbitrarily large subsets $A \subseteq X$, $|A| = n$, for which $-\log \delta_n = o(n)$. Let $x_i = 2^{-i^\alpha}$ and set

$$A = \{x_i + x_j : \ i = 1, \ldots, N, \ j = N+1, \ldots, 2N\},$$

so that $n = |A| = N^2$. We may assume that $\alpha \geq 1$ as the case $\alpha < 1$ can be handled with Falconer's criterion (1).

Observing that

$$x_i + x_j \leq x_{i'} + x_{j'} \iff (i,j) \geq (i',j') \ \text{ lexicographically},$$

it is not hard to see that

$$\delta_n \geq C\, 2^{-C\, N^\alpha},$$

which implies $-\log \delta_n \leq C N^\alpha = o(n)$, as we had to show. □

We should note that Falconer's criterion (1) does not apply even to the case $\alpha = 1$. One can prove that the sequence $\{2^{-n}\} + \{2^{-n}\}$ does not contain a subsequence $x_n \downarrow 0$ that satisfies (1).

3.5 Acknowledgment: I want to thank the referee for the careful reading and valuable comments.

4. Bibliography

[1] J. Bourgain, Construction of sets of positive measure not containing an affine image of a given infinite structure, Israel J. Math. **60** (1987), 3, 333-344.

[2] H.T. Croft, K.J. Falconer and R.K. Guy, *Unsolved Problems in Geometry*, Springer-Verlag, New York (1991).

[3] P. Erdős, My Scottish Book "problems", in *The Scottish Book*, R.D. Mauldin (ed.), Birkhäuser, Boston (1981).

[4] K.J. Falconer, On a problem of Erdős on sequences and measurable sets, Proc. Amer. Math. Soc. **90** (1984), 77-78.

[5] P. Komjáth, Large sets not containing images of a given sequence, Canad. Math. Bull. **26** (1983), 41-43.

[6] V. Lebedev and A. Olevskiĭ, Idempotents of Fourier multiplier algebra, Geom. Funct. Anal. **4** (1994), 5, 539–544.