Turán's extremal problem for positive definite functions on groups*

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Abstract

We study the following question: Given an open set Ω, symmetric about 0, and a continuous, integrable, positive definite function f, supported in Ω and with $f(0) = 1$, how large can $\int f$ be? This problem has been studied so far mostly for convex domains Ω in Euclidean space. In this paper we study the question in arbitrary locally compact abelian groups and for more general domains. Our emphasis is on finite groups as well as Euclidean spaces and $\mathbb{Z}^d$. We exhibit upper bounds for $\int f$ assuming geometric properties of Ω of two types: (a) packing properties of Ω and (b) spectral properties of Ω. Several examples and applications of the main theorems are shown. In particular we recover and extend several known results concerning convex domains in Euclidean space. Also, we investigate the question of estimating $\int_\Omega f$ over possibly dispersed sets solely in dependence of the given measure $m := |\Omega|$ of Ω. In this respect we show that in $\mathbb{R}$ and $\mathbb{Z}$ the integral is maximal for intervals.

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§0. Introduction

0.1 A problem of Turán and Stechkin

We study the following problem proposed by Turán and Stechkin [**St72**]:

> Given an open set Ω, symmetric about 0, and a continuous, positive definite, integrable function f, with $\operatorname{supp} f \subseteq \Omega$ and with $f(0) = 1$, how large can $\int f$ be?

The cases studied so far concern Ω being a convex subset of $\mathbb{R}^d$ [**AB01**, **AB02**, **Go01**, **KR03**] or an interval in the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ [**GM02**, **St72**, **AKP96**].

Such a question is interesting in the study of sphere packing [**Go00**, **CE03**], in additive number theory [**Rzs79**, **KMF78**] and in the theory of Dirichlet characters and exponential sums [**KS99**], among other things.

In this paper we study the problem in more general locally compact abelian (LCA) groups. This simplifies and unifies many of the existing results and gives several new estimates and examples. If G is a LCA group a continuous function $f \in L^1(G)$ is positive definite if its Fourier transform $\widehat{f} : \widehat{G} \to \mathbb{C}$ is everywhere nonnegative on the dual group $\widehat{G}$, see §1.1. For the relevant definitions of the Fourier transform we refer to [**Ka76**, Chapter VII] or [**Ru62**].

The set Ω will always be taken in this paper to be a 0-symmetric, open set in G.

If $f \in L^1(G)$ is continuous, positive definite and supported in Ω it follows that $f(0) \ge f(x)$ for any $x \in G$. This leads to the estimate $\int_G f \le |\Omega| f(0)$, called the *trivial estimate* from now on.

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Definition 1. *The* Turán constant $\mathcal{T}_G(\Omega)$ *of a* 0*-symmetric, open subset* Ω *of a LCA group* G *is the supremum of the quantity* $\int_G f/f(0)$*, where* $f \in L^1(G)$ *is continuous and positive definite, and* $\operatorname{supp} f$ *is a closed set contained in* Ω*.*

Remark 1. The quantity $\mathcal{T}_G(\Omega)$ depends on which normalization we use for the Haar measure on G. If G is discrete we use the counting measure and if G is compact and non-discrete we normalize the measure of G to be 1.

The *trivial upper estimate* for the Turán constant is $\mathcal{T}_G(\Omega) \le |\Omega|$.

0.2 Previous work

Let us review some of the known results.

Stechkin [**St72**] proves $\mathcal{T}_{\mathbb{T}}(\Omega) = \frac{1}{2}|\Omega|$ if $\Omega \subseteq \mathbb{T} = \mathbb{R}/\mathbb{Z}$ is a 0-symmetric interval whose half length divides the length of $\mathbb{T}$.

In [**AB01**, **AB02**] Arestov and Berdysheva prove that if $\Omega \subseteq \mathbb{R}^d$ is a convex polytope which can tile space when translated by the lattice $\Lambda \subseteq \mathbb{R}^d$ (this means that the copies $\Omega + \lambda$, $\lambda \in \Lambda$, are non-overlapping and almost every point in space is covered) then $\mathcal{T}_{\mathbb{R}^d}(\Omega) \le 2^{-d}|\Omega|$.

Gorbachev [**Go01**] also shows the same inequality if Ω is the Euclidean ball in $\mathbb{R}^d$ (a different proof of this is given in [**KR03**]). The ball clearly cannot tile space.

Kolountzakis and Révész [**KR03**] show the same inequality for all convex domains in $\mathbb{R}^d$ which are spectral (the definition appears later in this paper in §1.3)–convex spectral sets are conjecturally the same as convex tiles [**Fu74**]. It is known that all convex tiles are spectral (see e.g. [**KR03**]), so the result of Arestov and Berdysheva [**AB01**, **AB02**] is also a consequence of the result in [**KR03**].

Gorbachev and Manoshina [**GM02**] study the function $\mathcal{T}_{\mathbb{T}}(\Omega)$ when Ω is a 0-symmetric interval whose half length does not divide the length of $\mathbb{T}$, and they give more detailed information on $\mathcal{T}_{\mathbb{T}}(\Omega)$ when that length is of a certain arithmetical type.

0.3 Various forms of the Turán problem

In fact, it is worth noting that Turán type problems can be, and have been considered with various settings, although the relation of these has not been fully clarified yet. Thus in extending the investigation to LCA groups or to domains in Euclidean groups which are not convex, the issue of equivalence has to be dealt with. One may consider the following function classes.

$$\mathcal{F}_1(\Omega) := \left\{ f \in L^1(G) \ : \ \operatorname{supp} f \subset \Omega, \quad f \text{ positive definite} \right\}, \tag{1}$$

$$\mathcal{F}_{\&}(\Omega) := \left\{ f \in L^1(G) \cap C(G) \ : \ \operatorname{supp} f \subset \Omega, \quad f \text{ positive definite} \right\}, \tag{2}$$

$$\mathcal{F}_c(\Omega) := \left\{ f \in L^1(G) \ : \ \operatorname{supp} f \subset\subset \Omega, \quad f \text{ positive definite} \right\}, \tag{3}$$

$$\mathcal{F}(\Omega) := \left\{ f \in C(G) \ : \ \operatorname{supp} f \subset\subset \Omega, \quad f \text{ positive definite} \right\}. \tag{4}$$

In $\mathcal{F}_1, \mathcal{F}_{\&}$ $\operatorname{supp} f$ is assumed to be merely closed ad not necessarily compact, and in $\mathcal{F}_1, \mathcal{F}_c$ the function f may be discontinuous.

The respective Turán constants are

$$\begin{aligned} &\mathcal{T}_G^{(1)}(\Omega) \text{ or } \mathcal{T}_G^{\&}(\Omega) \text{ or } \mathcal{T}_G^{c}(\Omega) \text{ or } \mathcal{T}_G(\Omega) := \\ &\qquad \sup\left\{ \frac{\int_G f}{f(0)} \ : \ f \in \mathcal{F}_1(\Omega) \text{ or } \mathcal{F}_{\&}(\Omega) \text{ or } \mathcal{F}_c(\Omega) \text{ or } \mathcal{F}(\Omega), \text{ resp.} \right\}. \end{aligned} \tag{5}$$

In general we should consider functions $f : G \to \mathbb{C}$. But according to (10) also $\overline{f}$ and thus even $\varphi := \Re f$ is positive definite, while belonging to the same function class. As we also have $f(0) = \varphi(0)$ and $\int f = \int \varphi$, restriction to real valued functions does not change the values of the Turán constants.

To start with, we prove in §1.1

Theorem 1. *We have for any LCA group the equivalence of the above defined versions of the Turán constants:*

$$\mathcal{T}_G^{(1)}(\Omega) = \mathcal{T}_G^{\&}(\Omega) = \mathcal{T}_G^{c}(\Omega) = \mathcal{T}_G(\Omega)\,. \tag{6}$$

Note that the original formulation, presented also above in §0.1 corresponds to $\mathcal{T}_G^{\&}(\Omega)$.

Remark 2. It is not fully clarified what happens for functions vanishing only outside of Ω, but having nonzero values up to the boundary $\partial\Omega$.

0.4 New results

In this paper we focus mostly (but not exclusively) on finite or compact abelian groups.

Especially in the case of finite groups we can show clearly the geometric aspects of the problem without being sidetracked by technicalities that arise when the group is not discrete or not compact.

We present two types of results. In the first type some kind of "packing" condition is assumed on Ω which leads to an upper bound for $\mathcal{T}_G(\Omega)$. (The justification of the term "packing" should be more evident in the statement of Corollary 5 in §3.1.)

Theorem 2. *Suppose that G is a compact abelian group, $\Lambda \subseteq G$, $\Omega \subseteq G$ is a 0-symmetric open set and $(\Lambda - \Lambda) \cap \Omega \subseteq \{0\}$. Suppose also that $f \in L^1(G)$ is a continuous positive definite function supported on Ω. Then*

$$\int_G f(x)\,dx \le \frac{|G|}{|\Lambda|} f(0). \tag{7}$$

In other words $\mathcal{T}_G(\Omega) \le |G|/|\Lambda|$.

(Observe that the conditions imply that Λ is finite.)

The proof appears in §3.

The following Theorem 3 is analogous to Theorem 2 for the non-compact case.

Theorem 3. *Suppose that G is one of the groups $\mathbb{R}^d$ or $\mathbb{Z}^d$, that $\Lambda \subseteq G$ is a set of upper density $\rho > 0$, and $\Omega \subseteq G$ is a 0-symmetric open set such that $\Omega \cap (\Lambda - \Lambda) \subseteq \{0\}$. Let also $f \in L^1(G)$ be a continuous positive definite function on G whose support is a compact set contained in Ω. Then*

$$\int_G f(x)\,dx \le \frac{1}{\rho} f(0). \tag{8}$$

In other words $\mathcal{T}_G(\Omega) \le 1/\rho$.

In §2.4 and §3.2–§3.5 we present several examples and applications of Theorems 2 and 3 in various groups. These theorems in particular imply the results of [**St72**, **AB01**, **AB02**], but are much more general.

The second type of result we give is analogous to that proved in [**KR03**]. Here we suppose that Ω can be embedded in the difference set of a spectral set (see definition in §1.3) and we derive an upper bound for $\mathcal{T}_G(\Omega)$ from that.

Theorem 4. *Suppose G is a finite abelian group, $\Omega, H \subseteq G$, $\Omega \subseteq H - H$, and that H is a spectral set with spectrum $T \subseteq \widehat{G}$. Then for any positive definite function on G with support in Ω we have*

$$\sum_{x \in G} f(x) \le |H| f(0). \tag{9}$$

In other words $\mathcal{T}_G(\Omega) \le |H|$.

What was essentially proved in [**KR03**] was a "continuous" version of Theorem 4. Essentially, the following was proved.

Theorem 5. [**KR03**] *If H is a bounded open set in $\mathbb{R}^d$ which is spectral, then for the difference set $\Omega = H - H$ we have $\mathcal{T}_{\mathbb{R}^d}(\Omega) = |H|$.*

The result there was only formulated for convex sets $H \subset \mathbb{R}^d$ (for which $H - H = 2H$) but the proof works verbatim for the result we just stated. Let us emphasize here that in the continuous case we demand that eligible functions for our extremal problem have compact support contained in the given open set Ω (whose Turán constant we are estimating).

We give the proof of Theorem 4 in §4.1.

Furthermore, in §4.2 we show that there are cases when Theorems 4 and 5 give provably better results than any application of Theorems 2 and 3, respectively. For this we use one of Tao's [**Ta03**] recent examples which show one direction of Fuglede's conjecture to be false.

§1. Preliminaries

In this section we describe the basic facts about positive definite functions, translational tiling, packing and spectral sets on LCA groups.

1.1 Positive definite functions on LCA groups

In this subsection we explore a few facts on positive definite, not necessarily continuous functions. We could not decide if anything is new here, as we have found it very hard to locate these facts in the literature without assuming continuity of the positive definite function at the outset. So we collected these facts here.

Recall that on a LCA group G a function f is called positive definite if the inequality

$$\sum_{n,m=1}^{N} c_n \overline{c_m} f(x_n - x_m) \geq 0 \qquad (\forall x_1, \ldots, x_N \in G, \forall c_1, \ldots, c_N \in \mathbb{C}) \tag{10}$$

holds true. Note that positive definite functions are not assumed to be continuous. Still, all such functions f are necessarily bounded by $f(0)$ [**Ru62**, p. 18, Eqn (3)]. Moreover, $f(-x) = \widetilde{f}(x) := \overline{f(x)}$ for all $x \in G$ [**Ru62**, p. 18, Eqn (2)], hence the support of f is necessarily symmetric, and the condition $\operatorname{supp} f \subset \Omega$ implies also $\operatorname{supp} f \subset \Omega \cap (-\Omega)$. The latter set being symmetric, without loss of generality we can assume at the outset that Ω is symmetric itself.

It is immediate from (10) that for any subgroup K of G, the restriction $f|_K$ of a positive definite function f is also positive definite on K.

The Fourier transform $\widehat{f}$ of an $f \in L^1(G)$ belongs to $A(\widehat{G}) \subset C_0(\widehat{G})$, and the Fourier transform of the convolution $f * g$ of $f, g \in L^1(G)$, defined almost everywhere, satisfies $\widehat{f * g} = \widehat{f}\widehat{g}$ [**Ru62**, Theorem 1.2.4]. Similarly, for $\nu, \mu \in M(G)$ and their convolution $\mu * \nu \in M(G)$ the Fourier transforms are bounded and uniformly continuous and $\widehat{\mu * \nu} = \widehat{\mu}\widehat{\nu}$ [**Ru62**, Theorem 1.3.3].

In case $f, g \in L^2(G)$, the convolution $h := f * g$ is defined even in the pointwise sense and $h \in C_0(\widehat{G})$ [**Ru62**, Theorem 1.1.6(d)]. For $f \in L^2(G)$ arbitrary (denoting as above, $\widetilde{f}(x) := \overline{f(-x)}$), $f * \widetilde{f}$ is continuous and positive definite with Fourier transform $|\widehat{f}|^2$ [**Ru62**, 1.4.2(a)].

Note that for any given $\gamma \in \widehat{G}$ f is positive definite if and only if $f(x)\gamma(x)$ is positive definite; this can be checked by modifying the coefficients in (10) accordingly.

Lemma 1. *Suppose that f is (measurable and) positive definite and $g \in L^2(G)$ is arbitrary. Then the product $f \cdot (g * \widetilde{g})$ is positive definite.*

Proof. As written above, $h := g * \widetilde{g} \in C_0(G)$, while f, being positive definite, is also bounded. Take now $x_n \in G$ and $c_n \in \mathbb{C}$ for $n = 1, \ldots, N$ arbitrarily. Then

$$\begin{aligned} & \sum_{n,m=1}^{N} c_n \overline{c_m} f(x_n - x_m) h(x_n - x_m) \\ = \quad & \sum_{n,m=1}^{N} c_n \overline{c_m} f(x_n - x_m) \int_G g(x_n - y) \overline{g}(x_m - y) dy \\ = \quad & \int_G \sum_{n,m=1}^{N} a_n(y) \overline{a_m(y)} f(x_n - x_m) \; dy \,, \end{aligned}$$

where $a_n(y) := c_n g(x_n - y) \in L^2(G)$ $(n = 1, \ldots, N)$. Since the expression under the integral sign is nonnegative by (10) for each given y, also the integral is nonnegative and the assertion follows. $\square$

Note that we did not assume f to be integrable, and neither the product fh is supposed to belong to any subspace. By positive definiteness, f is bounded; but if G is not compact, $\widehat{f}$ is not necessarily defined. However, as $h \in C_0(G)$, in any case we must have $fh \in L^\infty(G)$. This follows from positive definiteness of fh, too.

The next Lemma is obvious for compact groups as we can take $k = 1$.

Lemma 2. *Suppose C is a compact set in a LCA group G and $\delta > 0$ is given. Then there exists a compactly supported, positive definite and continuous "kernel function" $k(x) \in C_c(G)$ satisfying $k(0) = 1$, $0 \le k \le 1$, and $k|_C \ge 1 - \delta$. Moreover, we can take $k = h * \widetilde{h}$, where h is the L^2-normalized indicator function of a suitable Borel measurable set $V \supset C$ with compact closure $\overline{V}$.*

Proof. We may clearly assume that G is not compact.

The deduction will follow the proof of 2.6.7 Theorem on page 52 of [**Ru62**] with a slight modification towards the end of the argument. In this proof the compact set C is given, and then another Borel set E and an increasing sequence of Borel sets $V_N \quad (N \in \mathbb{N})$ are found, so that $C \subset E = V_0$ and $|V_N| = (2N+1)^n |E|$ (with n a fixed nonnegative integer constant); moreover, all the V_N have compact closure and $V_N + E \subset V_{N+s}$ is ensured for some fixed s and for all $N \in \mathbb{N}$. Hence for every $c \in C \subset E$ we have $V_{N+s} - c \supset V_N$. Denoting the indicator function of V_{N+s} by χ we are led to $\int_G \chi(x+c)\chi(x)dx \ge |V_N|$. Putting $h := |V_{N+s}|^{-1/2}\chi$ yields $h * \widetilde{h}(c) \ge |V_N|/|V_{N+s}| > 1 - \delta$, if N is chosen large enough (depending on the constants n, s and the given δ). With this choice of h and $V := V_{N+s}$ all assertions of the Lemma are true. $\square$

Remark 3. As Rudin points out, this argument essentially depends on structure theorems of LCA groups.

Lemma 3. *Suppose that $f \in L^1(G)$ is positive definite. Then the Fourier transform $\widehat{f}$ is nonnegative.*

Proof. Since for any character $\gamma \in \widehat{G}$ we have $\widehat{\gamma f} = \widehat{f}(\cdot - \gamma)$, and f is positive definite precisely when γf is such, it suffices to prove that $\widehat{f}(0) \ge 0$.

For technical reasons, we need to modify f to have compact support. Let δ be any positive parameter. Since $d\nu(x) := f(x)dx$ is a *regular* Borel measure, for some compact set C we have $\|f\|_{L(G \setminus C)} < \delta$. Take the function k provided by Lemma 2 for the compact set C and the chosen parameter $\delta > 0$. If $g := kf$, Lemma 1 shows that g is positive definite, while $|\widehat{g}(0) - \widehat{f}(0)| \le |\int_C f - \int_C g| + \|f\|_{L^1(G\setminus C)} < \delta \int_C |f| + \delta \le \delta(1 + \|f\|_{L^1})$. Choosing δ small enough, it follows that there exists a compactly supported positive definite $g \in L^1(G)$ with $\widehat{g}(0) < 0$ provided that $\widehat{f}(0) < 0$. Hence it suffices to prove the assertion for compactly supported positive definite functions g.

Applying definition (10) with all c_n chosen as 1 yields

$$0 \le \sum_{n=1}^{N} \sum_{m=1}^{N} g(x_n - x_m)\,.$$

Integrating over C^N (where $C := \operatorname{supp} g$) we obtain

$$0 \le N|C|^N g(0) + (N^2 - N)|C|^{N-1} \int_C g\,,$$

which implies

$$-\frac{|C| g(0)}{N-1} \le \widehat{g}(0).$$

Letting $N \to \infty$ concludes the proof. $\square$

Lemma 4. *Suppose that $f, g \in L^1(G)$ are two positive definite functions. Then the convolution $f * g \in L^1(G)$ is uniformly continuous and positive definite.*

Proof. Since a positive definite function is bounded, we have also $f \in L^\infty(G)$, hence $f * g$ is uniformly continuous c.f. [**Ru62**, Theorem 1.1.6(b)]. For the Fourier transform $\widehat{f*g} = \widehat{f}\widehat{g}$ of the continuous function $f * g$ positive definiteness is equivalent to $\widehat{f*g} \geq 0$. Now Lemma 3 gives $\widehat{f} \geq 0$ and $\widehat{g} \geq 0$, hence $\widehat{f*g} \geq 0$ and $f * g$ is positive definite. ☐

Lemma 5. *Suppose U is a given neighborhood of 0 in a LCA group G. Then there exists a compactly supported, continuous, positive definite and nonnegative "kernel function" $k(x) \in C_c(G)$ satisfying* $\operatorname{supp} k \subset\subset U$ *and* $\int k = 1$. *Moreover, we can take $k = h * \widetilde{h}$, with $h = |W|^{-1}\chi_W$, where χ_W is the indicator function of a compact set W satisfying $W - W \subset U$.*

Proof. By continuity of the operation of subtraction, there exists a compact neighborhood W of 0 satisfying $W - W \subset U$. With the above definitions of k and h we clearly have $\operatorname{supp} k \subset\subset W - W \subset U$ (c.f. [**Ru62**, Theorem 1.1.6(c)] and also $\int k = |W|^{-2}(\int \chi_W)^2 = 1$. Since $h, \widetilde{h} \in L^2(G)$, $k \in C_0(G)$, and as $\operatorname{supp} k$ is compact, $k \in C_c(G)$. Since h is nonnegative, so is f. Finally, [**Ru62**, 1.4.2(a)] gives positive definiteness of k. ☐

Lemma 6. *Let f be positive definite and integrable. Then for any $\epsilon > 0$ and open set U containing 0, there exists a nonnegative, positive definite function of the form $k = h * \widetilde{h}$ (with $h \in L^2(G)$), so that* $\operatorname{supp} k \subset\subset U$, $\int_U k = 1$, *and* $\|f - f * k\|_1 < \epsilon$.

Proof. For the given function f there exists a neighborhood V of 0 with the property that $\|f - f * u\| < \epsilon$ whenever $\int_G u = 1$ and $u \geq 0$ is Borel measurable and vanishing outside V [**Ru62**, Theorem 1.1.8]. Now we can construct for the open set $U_0 := V \cap U$ the kernel function k as in Lemma 5. Clearly, k satisfies all conditions for u, hence $\|f - f * k\|_1 < \epsilon$ follows. By construction, $\operatorname{supp} k \subset\subset U_0 \subset U$ and $\int_U k = 1$. ☐

Lemma 7. *For any pair of sets $K \subset\subset U$ with K compact and U open, there exists a neighborhood V of 0 satisfying $K + V \subset U$.*

Proof. Since addition is continuous, for any open neighborhood U_0 of 0 there exists a neighborhood W so that $W + W \subset U_0$. Take now to each point $x \in K$ an open neighborhood W_x of 0 such that $x + W_x + W_x \subset U$, ie. $W_x + W_x \subset U - x$. Clearly the family of open sets $\{x + W_x \ : \ x \in K\}$ form an open covering of K, so in view of compactness of K there exists a finite subcovering $\{W_{x_k} + x_k \ : \ k = 1, \dots, n\}$. Take now $V := \bigcap_{k=1}^n W_{x_k}$. We claim that $K + V \subset U$. Indeed, if $y \in K$ and $z \in V$ then considering any index k with $y \in x_k + W_{x_k}$, we find $y + z \in (x_k + W_{x_k}) + V \subset x_k + W_{x_k} + W_{x_k} \subset U$. ☐

Lemma 8. *Let $\epsilon > 0$ be arbitrary. Assume that f is measurable and positive definite and compactly supported in the open set Ω. Then there exists another positive definite, but also continuous function g with $f(0) \geq g(0)$ and $\int_G g \geq \int_G f - \epsilon$, also supported compactly in Ω.*

Proof. Observe that f, being positive definite, is also bounded, and since it is compactly supported, it also belongs to $L^1(G)$. Thus we can use the Fourier transform $\widehat{f}$. Let $K := \operatorname{supp} f \subset\subset \Omega$ and consider a neighborhood U of 0 with $K + U \subset \Omega$. Such a U is provided by Lemma 7. Lemma 6 provides a positive definite, continuous kernel $k \in C_c(G)$, compactly supported in U and satisfying $\int_G g \geq \int_G f - \epsilon$. In view of $k = h * \widetilde{h}$ and Lemma 4 also $g := f * k$ is positive definite while obviously $g \in C_c(G)$ is supported compactly in $K + U \subset \Omega$. It remains to note that by $k \geq 0$, $\int k = 1$ and $|f| \leq f(0)$ we also have $g(0) = \int k(x) f(-x) dx \leq f(0) \int k = f(0)$. ☐

Proposition 1. *With the definitions above we have $\mathcal{T}_G^{(1)}(\Omega) = \mathcal{T}_G^c(\Omega)$.*

Proof. Let $\epsilon > 0$ and $\delta > 0$ be arbitrary and $f \in \mathcal{F}_1(\Omega)$ be chosen so that $\int_G f > \mathcal{T}_G^{(1)}(\Omega) - \delta$. As $f \in L^1(G)$, the measure $|f(x)|dx$ is absolutely continuous with respect to the Haar measure, hence it is also a regular Borel measure and there exists a compact subset $C \subset\subset \operatorname{supp} f$ so that $\int_{G \setminus C} |f| < \delta$. Now an application of Lemma 2 with C and δ provides us the positive definite, compactly supported kernel function k satisfying $k(0) = 1$, and $k|_C > (1 - \delta)$. Let $g := fk$. Then $\operatorname{supp} g \subset (\operatorname{supp} k \cap \operatorname{supp} f) \subset\subset \operatorname{supp} f$, hence g is compactly supported within Ω. Moreover, $g(0) = 1$ and g is positive definite in view of Lemma 1. Hence $g \in \mathcal{F}_c(\Omega)$. We now have

$$\begin{aligned}\int g = \int_\Omega kf &= \int_\Omega f - \int_\Omega (1-k)f \\ &\geq \int_\Omega f - \delta \int_C |f| - \int_{\Omega\setminus C} |f| \\ &\geq \int_\Omega f - \delta \int_\Omega |f| - \delta \qquad \geq (1-\delta)\big(\mathcal{T}_G^{(1)}(\Omega) - \delta\big) - \delta.\end{aligned}$$

Clearly, if δ was chosen small enough, we obtain $\int g > \mathcal{T}_G^{(1)}(\Omega) - \epsilon$. Now taking sup over $g \in \mathcal{F}_c(\Omega)$ concludes the proof, since $\epsilon > 0$ was arbitrary. □

Proposition 2. *With the definitions above we have $\mathcal{T}_G(\Omega) = \mathcal{T}_G^c(\Omega)$.*

Proof. Since $\mathcal{F}_c(\Omega) \supset \mathcal{F}(\Omega)$, it suffices to prove $\mathcal{T}_G^c(\Omega) \leq \mathcal{T}_G(\Omega)$.

Let $\epsilon > 0$ and $f \in \mathcal{F}_c(\Omega)$ be chosen so that $\int f > \mathcal{T}_G^c(\Omega) - \epsilon$, while $\operatorname{supp} f$ is a compact subset of the open set Ω. Hence an application of Lemma 8 provides a $g \in \mathcal{F}(\Omega)$ with $\mathcal{T}_G(\Omega) \geq \int g > \int f - \epsilon > \mathcal{T}_G^c(\Omega) - 2\epsilon$. Now $\epsilon \to 0$ yields the Proposition. □

Proof of Theorem 1. We have the obvious inclusions $\mathcal{F}_1(\Omega) \supset \mathcal{F}_\&(\Omega) \supset \mathcal{F}(\Omega)$ and $\mathcal{F}_1(\Omega) \supset \mathcal{F}_c(\Omega) \supset \mathcal{F}(\Omega)$, hence $\mathcal{T}_G^1(\Omega) \geq \mathcal{T}_G^\&(\Omega) \geq \mathcal{T}_G(\Omega)$ and $\mathcal{T}_G^{(1)}(\Omega) \geq \mathcal{T}_G^c(\Omega) \geq \mathcal{T}_G(\Omega)$. On combining these inequalities with Propositions 1 and 2 the assertion follows. □

If we consider a *continuous* positive definite function f, then it must also be uniformly continuous [**Ru62**, p. 18, Eqns (3), (4)]. When $\operatorname{supp} f$ has bounded Haar measure (and, in particular, when $\operatorname{supp} f$ is compact) then f belongs to $L^1(G)$, too. For an integrable, continuous and positive definite function f the Fourier transform $\widehat{f}$ of f exists, and the Fourier inversion formula holds, cf. [**Ru62**, 1.5.1]. The well-known Bochner-Weil characterization says that $f \in C(G)$ being positive definite is equivalent to the existence of a non-negative measure μ on the dual group $\widehat{G}$ so that

$$f(x) = \int_{\widehat{G}} \overline{\gamma(x)}\, d\mu(\gamma);$$

moreover, this representation is unique cf. [**Ru62**, 1.4.3], Comparing the Fourier inversion formula and the unique representation above leads to the further characterization that *for a continuous and integrable f* being positive definite is equivalent to $\widehat{f} \geq 0$, compare [**Ru62**, 1.7.3(e)]. Thus it is really advantageous to restrict the function class considered from $\mathcal{F}_1(\Omega)$ to $\mathcal{F}(\Omega)$, say.

Our setting is that Ω is an open (symmetric) set, and we require that f can be nonzero only in Ω. This is an essential condition. In this respect approximation has its limitations: eg. we cannot relax the conditions to require $\operatorname{supp} f \subset \overline{\Omega}$ only.

Indeed, if Ω is not *fat*, meaning that $\Omega = \operatorname{int} \overline{\Omega}$, this can lead to essential changes of the Turán constants. Eg. if $G = \mathbb{R}$ and $\Omega = (-a, a) \setminus \{\pm b\}$, then $\operatorname{int}\overline{\Omega} = (-a, a)$ and $\mathcal{T}_\mathbb{R}((-a,a)) = a$, while $\mathcal{T}_\mathbb{R}(\Omega) = b$ if $a/2 \leq b \leq a$, see Theorem 7 below. Similarly, if $G = \mathbb{T}$ and $\Omega = \mathbb{T} \setminus \{\pi\}$, then $\mathcal{T}_\mathbb{T}(\Omega) = 1/2$, but obviously $\mathcal{T}_\mathbb{T}(\overline{\Omega}) = 1$. That is, forcing the function f to vanish at one single point can, through positive definiteness, bring down the values essentially in general.

In this respect, original formulations of the Turán problem in [**AB02**] and [**KR03**] may be misleading, since for a convex body Ω in $\mathbb{R}^d$ or $\mathbb{T}^d$ the allegedly extremal function $\chi_{\Omega/2} * \chi_{\Omega/2}$ does *not* belong to the function class $\mathcal{F}_\&(\Omega)$ considered there. Instead, a corresponding limiting argument should provide the same extremal value. In convex or star bodies in Euclidean spaces one can easily obtain a positive definite function supported properly in the body from one that may be "non-zero up to the boundary", by a slight dilation of space, without losing much integral. It is unclear how to do this in general, even for domains in $\mathbb{R}^d$.

1.2 Tiling and packing

Suppose G is a LCA group. We say that a function $f \in L^1(G)$ tiles G by translation with a set $\Lambda \subseteq G$ at level $c \in \mathbb{C}$ if
$$\sum_{\lambda\in\Lambda} f(x-\lambda) = c$$
for a.a. $x \in G$, with the sum converging absolutely. We then write "$f + \Lambda = cG$".

We say that f *packs* G with the translation set Λ at level $c \in \mathbb{R}$, and write $f + \Lambda \le cG$, if
$$\sum_{\lambda\in\Lambda} f(x-\lambda) \le c,$$
for a.a. $x \in G$.

When the group is finite (and we do not, therefore, have to worry about the set Λ being finite or not) the tiling condition $f + \Lambda = cG$ means precisely $f * \chi_\Lambda = c$. Taking Fourier transform, this is the same as $\widehat{f}\widehat{\chi_\Lambda} = c|G|\chi_{\{0\}}$, which is in turn equivalent to the condition
$$\operatorname{supp} \widehat{\chi_\Lambda} \subseteq \{0\} \cup \left\{\widehat{f} = 0\right\} \ \text{ and } \ c = \frac{|\Lambda|}{|G|}\sum_{x\in G} f(x). \tag{11}$$

Finally, if $E \subseteq G$ we say that E packs with Λ if χ_E packs with Λ at level 1. Observe that E packs with Λ if and only if
$$(E-E)\cap(\Lambda-\Lambda) = \{0\}.$$

1.3 Spectra

Let G be a LCA group and $\widehat{G}$ be its dual group, that is the group of all continuous group homomorphisms $G \to \mathbb{C}$. A set $H \subseteq G$ has the set $T \subseteq \widehat{G}$ as a *spectrum* if and only if T forms an orthogonal basis for $L^2(H)$.

Suppose from now on that G is finite.

It follows that $|T| = |H|$, the dimension of $\ell^2(H)$, and with a little more work it follows that T is a spectrum of H if and only if we have the tiling condition
$$\left|\widehat{\chi_H}\right|^2 + T = |H|^2\widehat{G}. \tag{12}$$

Indeed, for $t_1, t_2 \in \widehat{G}$ we have by definition of the Fourier transform that
$$\langle t_1, t_2\rangle_H = \sum_{x\in H} t_1(x)\overline{t_2}(x) = \sum_{x\in H}(t_1 - t_2)(x) = \widehat{\chi_H}(t_1 - t_2).$$

Suppose now that T is a spectrum of H. If $t \in \widehat{G}$ we have (Parseval)
$$\begin{aligned} |H| &= \|t\|^2_{\ell^2(H)} \\ &= \sum_{s\in T}\left|\left\langle t, \frac{s}{\|s\|}\right\rangle\right|^2 \\ &= \frac{1}{|H|}\sum_{s\in T}|\langle t, s\rangle|^2 \\ &= \frac{1}{|H|}\sum_{s\in T}\left|\widehat{\chi_H}(t-s)\right|^2, \end{aligned}$$
which is precisely the statement that $\left|\widehat{\chi_H}\right|^2 + T = |H|^2\widehat{G}$. That this tiling condition is also sufficient to imply that T is a spectrum of H follows similarly (we are not using this direction in this paper).

By the analysis of tiling shown in §1.2 it follows that this happens if and only if
$$\operatorname{supp} \widehat{\chi_T} \subseteq \{0\} \cup (H-H)^c \ \text{ and } \ |T| = |H|. \tag{13}$$

§2. Generalities about Turán constants on groups

2.1 Homomorphic images and the Turán constant

Let G and H be two LCA groups, and $\varphi : G \to H$ a continuous group homomorphism which maps G onto H. Denote $K := \mathrm{Ker}(\varphi) \le G$. By continuity of φ, K is a closed subgroup, hence a LCA group itself. We consider G/K as fixed together with the canonical or natural projection $\pi : G \to G/K$ defined as $\pi(g) := [g] := g + K \in G/K$. By definition of the topology of G/K, π is an open and continuous mapping. Compare §B.2, B.6 in Appendix B of [**Ru62**]. Moreover, $\varphi \circ \pi^{-1} : G/K \to H$ is an isomorphism of the LCA groups G/K and H.

For the determination of the Turán constants, the choice of the Haar measure is relevant. Haar measures are unique up to a constant factor: we can always choose the Haar measures μ_K and $\mu_{G/K}$ so that $d\mu_G = d\mu_K d\mu_{G/K}$, in the sense of (2) in [**Ru62**, 2.7.3]. On the other hand fixing a particular Haar measure μ_H of H always leaves open the question of compatibility with the fixed measure $\mu_{G/K}$ and the mapping φ. Let $A \subset H$ be an arbitrary Borel set. Then one can define $\nu(A) := \mu_{G/K}(\pi(\varphi^{-1}(A)))$; since φ is onto, clearly this defines another Haar measure on H. Since Haar measures are constant multiples of each other, we necessarily have $C := d\mu_H/d\nu$ a constant. Once H and μ_H are given, various homomorphisms φ may generate different measures, but the constant $C = C(\varphi)$ can always be read from this relation.

Proposition 3. *Let G and H be LCA groups, and $\varphi : G \to H$ be a continuous group homomorphism onto H. Suppose an open subset $\Omega \subset G$ is given, and let $\Theta := \varphi(\Omega) \subset H$. Consider the closed subgroup $K := \mathrm{Ker}(\varphi) \le G$, and the quotient group G/K together with their Haar measures $\mu_{G/K}$ and μ_K, normalized as above. We then have*

$$\mathcal{T}_G(\Omega) \le \frac{1}{C}\mathcal{T}_H(\Theta)\mathcal{T}_K(\Omega \cap K) \qquad (C := \frac{d\mu_H}{d\nu}) \quad . \tag{14}$$

Here $\nu := \mu_{G/K} \circ \pi \circ \varphi^{-1}$ is defined as above.

Proof. As K is the kernel of the continuous homomorphism φ, K is a closed subgroup of G. Therefore, the factor group G/K is a LCA group, which is continuously isomorphic to H.

The image Θ of the open set Ω is open, since φ is also an open mapping. Indeed, π is open by its definition, and thus $\pi(\Omega)$ is open in G/K for any open Ω in G. However, the isomorphism $\psi : G/K \to H$, defined by $\psi := \varphi \circ \pi^{-1}$, brings over the open set $\pi(\Omega)$ to Θ, which is then open by the isomorphism itself.

Observe that $\Omega_g := \Omega \cap (K + g)$ is relatively open for any $g \in G$, while the coset $K + g$ is closed. Let us choose arbitrarily a representative $g(h) \in G$ of each coset $\varphi^{-1}(h)$ of K to all $h \in H$. Now for any uniformly continuous function $f : G \to \mathbb{C}$ we can define

$$F(h) := \int_K f(g(h) + k)\ d\mu_K(k) = \int_{\varphi^{-1}(h)} f(x)\ d\mu_K(x - g(h)). \tag{15}$$

Since f is uniformly continuous, the function $F : H \to \mathbb{C}$ is continuous, $F(0) = \int_K f d\mu_K$, and by Fubini's Theorem

$$\begin{aligned} \int_H F(h) d\mu_H(h) &= \int_H \int_{\varphi^{-1}(h)} f(g(h)+k)\ d\mu_K(k) C d\nu(h) \\ &= C \int_H \int_K f(g(h)+k)\ d\mu_K(k) d\mu_{G/K}(\pi\varphi^{-1}(h)) \\ &= C \int_{H \times K} f(g(h)+k)\ d\mu_K(k) d\mu_{G/K}([g(h)]) = C \int_G f\ d\mu_G \,, \end{aligned} \tag{16}$$

taking into account the choice of normalization of the Haar measures for K and G/K. Next we prove that F is positive definite on H in case f is positive definite on G. Indeed, for any character χ on H there is a character $\gamma := \chi \circ \varphi$ on G, and applying (16) to $f\gamma$ yields

$$\int_H F(h)\chi(h) d\mu_H(h) = C \int_G f(g)\gamma(g) \ge 0 \ .$$

Thus we have $\int_H F d\mu_H \le \mathcal{T}_H(\Theta) F(0)$. Moreover, $f|_K$ is positive definite on K, hence we also have $F(0) = \int_{K \cap \Omega} f\ d\mu_K \le \mathcal{T}_K(K \cap \Omega) f(0)$. Comparing these inequalities with (16) yields $C \int_G f\ d\mu_G \le \mathcal{T}_H(\Theta)\mathcal{T}_K(K \cap \Omega) f(0)$, and taking supremum of $\int_G f d\mu_G / f(0)$ (14) obtains. □

2.2 Automorphic invariance of the Turán constant

One of the reasons to work out Proposition 3 is its corollary to the case when we deal with an automorphism of the group G.

Corollary 1. *Let G be a LCA group and let $\varphi : G \to G$ be an automorphism. Then we have for any open set $\Omega \subset G$ the identity*

$$\mathcal{T}_G(\varphi(\Omega)) = \frac{|\varphi(\Omega)|}{|\Omega|}\mathcal{T}_G(\Omega) \ . \tag{17}$$

Proof. In our case $H = G$ and φ is an automorphism. Clearly then $K = \{0\}$ is the trivial group, $\mu_K = \delta_0$ is the trivial measure, $K \cap \Omega = \{0\}$, $\mathcal{T}_K(K \cap \Omega) = 1$, $\mu_K(K \cap \Omega) = 1$ and $G/K \cong G$, $\mu_{G/K} \cong \mu_G$. Thus we find $\nu = \mu_G \circ \varphi^{-1}$, and $C := d\mu_H/d\nu$ being constant, it can be computed on $\Omega^* := \varphi(\Omega)$ as $C = |\varphi^{-1}(\varphi(\Omega))|/|\varphi(\Omega)| = |\Omega|/|\varphi(\Omega)|$. Applying Proposition 3 yields (17) with $\leq$ first. However, φ^{-1} is also an automorphism, and that implies the reverse inequality, too. Whence Corollary 1 follows. $\square$

The important special case when $G = \mathbb{R}^d$ and φ is any linear mapping $A : \mathbb{R}^d \to \mathbb{R}^d$ was already noted in [**AB01**]. There the computation of the constant C is equivalent to the calculation of the volume element, ie. the determinant, of the linear mapping A.

The next assertion was also observed in [**AB01**] for $\mathbb{R}^d$.

Corollary 2. *Let $G = G_1 \times \cdots \times G_n$ and $\Omega_j \subset G_j$ $(j = 1, \ldots, n)$, $\Omega = \Omega_1 \times \cdots \times \Omega_n$. Then we have*

$$\mathcal{T}_G(\Omega) = \mathcal{T}_{G_1}(\Omega_1) \cdots \mathcal{T}_{G_n}(\Omega_n) \ . \tag{18}$$

Proof. The $\leq$ direction easily follows from iteration of Proposition 3. On the other hand take any positive definite functions f_j on G_j with $\operatorname{supp} f_j \subset\subset \Omega_j$ for $(j = 1, \ldots, n)$. It is easy to see that then the product $f := f_1 \cdots f_n$ is a positive definite function on G, with $\operatorname{supp} f \subset\subset \Omega$, hence also the $\geq$ part of (18) follows. $\square$

2.3 Turán constants on quotient groups

Corollary 3. *Let G be a LCA group, K a closed subgroup of G, and suppose that the Haar measures μ_K and $\mu_{G/K}$ of G and G/K, respectively, are normalized (as always) so that $d\mu_G = d\mu_K d\mu_{G/K}$. Let Ω be any open set in G and Θ be its projection on G/K, ie. $\Theta := \{g + K \ : \ g \in \Omega\}$. Then we have*

$$\mathcal{T}_G(\Omega) \leq \mathcal{T}_{G/K}(\Theta)\mathcal{T}_K(\Omega \cap K) \quad . \tag{19}$$

In particular, if $\Omega \cap K = \{0\}$, then $\mathcal{T}_G(\Omega) \leq \mathcal{T}_{G/K}(\Theta)$.

Proof. Consider $H := G/K$ and the natural projection $\pi : G \to G/K$. It is a continuous group homomorphism and thus Proposition 3 can be applied with $\varphi := \pi$. In this case $\Theta = \pi(\Omega)$ comprises the class of cosets $K + g$ so that $K + g \cap \Omega \neq \emptyset$, and the arising measure ν is identical to $\mu_{G/K}$. Hence $C = 1$ and we are led to (19). The special case is obvious. $\square$

2.4 Restrictions to subgroups and the Turán constants

We show now that there is some sort of monotonicity in the first argument of $\mathcal{T}_G(\Omega)$ as well.

Corollary 4. *Let G be a compact abelian group, and K a closed subgroup of G. Let the Haar measures μ_K and μ_G be normalized arbitrarily, and let Ω be any open set in G. Then we have*

$$\mathcal{T}_G(\Omega) \leq \frac{|G|}{|K|}\mathcal{T}_K(\Omega \cap K) \quad . \tag{20}$$

Here $|G| = \mu_G(G)$ and $|K| = \mu_K(K)$.

Proof. With μ_G and μ_K already given, we can define the Haar measure $\mu_{G/K}$ so that the condition $\mu_G = \mu_K\mu_{G/K}$ still holds. Let $\varphi := \pi$ and $H := G/K$ as in the previous Corollary. Since we always have $\Theta \subset G/K$, and thus $\mathcal{T}_{G/K}(\Theta) \leq \mathcal{T}_{G/K}(G/K) = \mu_{G/K}(G/K)$, an application of Corollary 3 yields $\mathcal{T}_G(\Omega) \leq \mu_{G/K}(G/K)\mathcal{T}_K(\Omega \cap K)$. It remains to see that for a *compact* group G and (closed, hence compact) subgroup K also the quotient is compact, and according to our choice of normalization we have $\mu_{G/K}(G/K) = \mu_G(G)/\mu_K(K)$. The assertion follows. $\square$

Example 1. Let us remark here that Lemma 1 of [**GM02**] can be proved via Corollary 4 by taking $G = \mathbb{T}$, Ω to be the interval $(-p/q, p/q) \subseteq \mathbb{T}$ (for some co-prime integers p and q with $p/q \le 1/2$) and K to be the (finite) subgroup of $\mathbb{T}$ generated by $1/q$. The results in [**GM02**] first show that the Turán problem in this case can be reduced to a finite problem of linear programming (this is obviously the case for any Turán problem on a finite group) and Corollary 4 shows half the reduction. The reverse inequality is also true in this particular case (this can be shown by "convolving" a positive definite function on the subgroup with a Fejér kernel of half-base $1/q$) but it cannot be expected to hold in general.

§3. Upper bound from packing

Here we show in §3.1 the three main results which give us upper bounds for the Turán constant using "packing". In the remaining part of this section we show several examples and applications of these, in various groups.

3.1 Proof of the main bounds from "packing"

Proof of Theorem 2. Define $F : G \to \mathbb{C}$ by

$$F(x) = \sum_{\lambda,\mu\in\Lambda} f(x + \lambda - \mu).$$

In other words $F = f * \delta_\Lambda * \delta_{-\Lambda}$, where δ_A denotes the finite measure on G that assigns a unit mass to each point of the finite set A. It follows that $\widehat{F} = \widehat{f}\left|\widehat{\delta_\Lambda}\right|^2 \ge 0$ so that F is continuous and positive definite. Moreover, we also have

$$\operatorname{supp} F \subseteq \operatorname{supp} f + (\Lambda - \Lambda) \subseteq \Omega + (\Lambda - \Lambda) \tag{21}$$

and

$$F(0) = |\Lambda| f(0), \tag{22}$$

since $\Omega \cap (\Lambda - \Lambda) \subseteq \{0\}$. Finally

$$\int_G F = |\Lambda|^2 \int_G f. \tag{23}$$

Applying the trivial upper bound $\int_G F \le F(0)|\Omega + (\Lambda - \Lambda)|$ to the positive definite function F and using (22) and (23) we get

$$\int_G f \le \frac{|\Omega + (\Lambda - \Lambda)|}{|\Lambda|} f(0). \tag{24}$$

Estimating trivially $|\Omega + (\Lambda - \Lambda)|$ from above by $|G|$ we obtain the required $\mathcal{T}_G(\Omega) \le |G|/|\Lambda|$.
□

Corollary 5. *Let G be a compact abelian group and suppose $\Omega, H, \Lambda \subseteq G$, $H + \Lambda \le G$ is a packing at level 1, that $\Omega \subseteq H - H$ and that $f \in \mathcal{F}(\Omega)$. Then (7) holds.*

In particular, if $H + \Lambda = G$ is a tiling, we have

$$\mathcal{T}_G(\Omega) \le |H|. \tag{25}$$

Proof. Since $H + \Lambda \le G$ it follows that $(H - H) \cap (\Lambda - \Lambda) = \{0\}$. Since $\Omega \subseteq H - H$ by assumption it follows that Ω and $\Lambda - \Lambda$ have at most 0 in common. Theorem 2 therefore applies and gives the result. If $H + \Lambda = G$ then $|G|/|\Lambda| = |H|$ and this proves (25). □

Proof of Theorem 3. Let $\epsilon > 0$ and choose $R > 0$ and $x \in G$ such that

$$|\Lambda \cap Q_R(x)| \ge (\rho - \epsilon)|Q_R(x)| \ge (\rho - \epsilon)(R - 1)^d,$$

where $Q_R(x)$ is the cube of side R and center at x. Assume also that $\operatorname{supp} f \subseteq Q_r(0)$.

Let $\Lambda' = \Lambda \cap Q_R(x)$ and construct the function F as in the proof of Theorem 2, with Λ' in place of Λ. We now have that

$$\operatorname{supp} F \subseteq \operatorname{supp} f + (\Lambda' - \Lambda') \subseteq Q_{2R+r}(0).$$

This time we do not apply the trivial upper estimate to F as we did in Theorem 2 (then, we had no detailed information on the support). Instead we use that for $L \in 2\mathbb{N}$

$$\mathcal{T}_G(Q_L(0)) \leq (L/2+1)^d \ . \tag{26}$$

The validity of $\mathcal{T}_{\mathbb{R}^d}(Q_L(0)) \leq 2^{-d}L^d$ ($\forall L > 0$) and hence (26) in the case of $G = \mathbb{R}^d$ has been proved, for example, in [**AB01**, **AB02**, **KR03**]. For $G = \mathbb{Z}^d$ we give a proof here.

Notice first that for any finite $\Omega \subseteq \mathbb{Z}^d$ and any large enough positive integer M we have

$$\mathcal{T}_{\mathbb{Z}^d}(\Omega) \leq \mathcal{T}_{\mathbb{Z}^d_M}(\Omega). \tag{27}$$

Indeed, if M is large enough (e.g. $M > \mathrm{diam}(\Omega)/2$) then the closed subgroup $K := M\mathbb{Z}^d$ only intersects Ω in 0, while the factor group $\mathbb{Z}^d_M$ will have an injective image Θ of Ω: hence Corollary 3 applies.

If $\Omega = Q^d_L(0) = \{-L/2, \ldots, L/2\}^d$ define H to be the set $\{0, \ldots, L/2\}^d$ such that $\Omega = H - H$. Take now $M = 10(L/2+1)$, for example, so that (a) H tiles $\mathbb{Z}^d_M$ by translation, and, (b) M is large enough to have all elements of Ω distinct mod $\mathbb{Z}^d_M$. Using Corollary 5 we obtain (26) from (25) in the group $\mathbb{Z}^d_M$, and hence also in $\mathbb{Z}^d$ because of (27).

Hence taking $L := L(R, r)$ in (26) as the least even integer not less than $2R + r$, we obtain both for $G = \mathbb{R}^d$ and $G = \mathbb{Z}^d$ the estimate $\int_G F \leq \mathcal{T}_G(Q_L(0))F(0) \leq (R + r/2 + 2)^d F(0)$. Comparing this with (22) and (23) (with Λ' in place of Λ) we are led to

$$\int_G f \leq f(0)\frac{(R+r/2+2)^d}{|\Lambda'|} \leq \frac{(R+r/2+2)^d}{(\rho-\epsilon)R^d} \ .$$

Since $\epsilon > 0$ can be taken arbitrarily small and R arbitrarily large, we get $\int_G f \leq \frac{1}{\rho} f(0)$. $\square$

3.2 Sharpness

The bound (7) can be sharp. Take, for example, Ω to be a subgroup of G of finite index and $H = \Omega$. Take also Λ to a complete set of coset representatives of G/Ω, so that $|\Lambda| < \infty$. Then $H + \Lambda = G$ and Corollary 5 applies and gives

$$\sum_{x \in G} f(x) \leq |\Omega| f(0) \tag{28}$$

for every positive definite function $f : G \to \mathbb{C}$ supported in Ω, which is also the trivial bound. Taking $f = \chi_\Omega$, which is positive definite because Ω is a group, gives equality in (28).

More generally (and as in the next example) the inequality (7) is sharp whenever $H + \Lambda = G$ and $\Omega = H - H$. In such a case the function $f = \chi_H * \chi_{-H}$ achieves equality in (7).

3.3 Examples

Example 2. Take $G = \mathbb{Z}_8 = \{0, 1, \ldots, 7\}$, $H = \{0, 1, 4, 5\}$, $\Omega = H - H = \{0, 1, 3, 4, 5, 7\}$ and $\Lambda = \{0, 2\}$, so that $\Lambda - \Lambda = \{0, 2, 6\}$ and $H + \Lambda = G$. It follows that

$$\sum_{x \in G} f(x) \leq 4f(0)$$

for any positive definite function on $\mathbb{Z}_8$ which vanishes on ± 2, instead of the trivial $\sum_{x \in G} f(x) \leq 6f(0)$. The equality can be achieved by the function $f = \chi_H * \chi_{-H}$.

Example 3. Let $G := \mathbb{Z}$ and $\Omega := \Omega_N := \{-N, -1, 0, 1, N\}$; then the trivial estimate is $A(N) := \mathcal{T}_{\mathbb{Z}}(\Omega_N) \leq 5$. Let $f \in \mathcal{F}(\Omega)$ be a positive definite and real valued function: then $f(k) = f(-k)$, that is, f is even. The dual group is $\mathbb{T}$, and positive definiteness of f means $p(x) := 1 + 2f(1)\cos x + 2f(N)\cos Nx \geq 0$ (as $f(0) = 1$ by normalization). In the Turán problem we are to maximize $\int_{\mathbb{Z}} f = 1 + 2f(1) + 2f(N) = p(0)$; we have $A(N) = \max p(0)$.

To find $A(N)$ in case when $N = 2n+1$ is odd we may look at the value $p(\pi) = 1 - 2f(1) - 2f(2n+1) \geq 0$ to see that $p(0) = 2 - p(\pi) \leq 2$. Clearly, any function with $f(1) + f(2n+1) = 1/2$ achieves this bound while $p \geq 0$ if additionally we require $0 \leq f(1), f(2n+1)$. Hence $A(2n+1) = 2$.

If $N=2n$ is even, the solution is less simple. We claim that $A(N)=1+1/\cos\frac{\pi}{2n+1} =: C(N)$, say, and the extremal function is

$$p_0(x) := 1 + \frac{2n}{(2n+1)\cos\frac{\pi}{2n+1}}\cos x + \frac{1}{(2n+1)\cos\frac{\pi}{2n+1}}\cos 2nx\,.$$

Clearly $p_0(0)=C(N)$, and standard calculus proves nonnegativity of p_0, hence it is an admissible trigonometric polynomial and $A(N)\geq C(N)$.

To show its extremality we consider a general $p(x)=1+a\cos x+b\cos 2nx$ (where $a:=2f(1)$, $b:=2f(N)$) at the point $z_0:=\pi+\pi/(2n+1)$, which yields $0\leq p(z_0)=1-a\cos\frac{\pi}{2n+1}-b\cos\frac{\pi}{2n+1}$. Thus $p(0)=1+a+b=1+(1-p(z_0))/\cos\frac{\pi}{2n+1}\leq C(N)$, and the calculation is concluded.

Now let us consider the estimates obtainable from the use of Theorem 3. In case N is odd, taking $\Lambda:=2\mathbb{Z}$ is optimal. Indeed, since Λ is a subgroup, $\Lambda-\Lambda=\Lambda$, and it does not intersect Ω_N (apart from 0), hence an application of Theorem 3 gives the right value $A(N)\leq 1/\mathrm{dens}(\Lambda)=2$. Hence in this case Theorem 3 is sharp.

Let us see that it is *not* in the case when $N=2n$ is even. To this, first we have to find the best upper density, that is,

$$L(N) := \sup_{\Omega_N\cap(\Lambda-\Lambda)=\{0\}} \overline{\mathrm{dens}}(\Lambda)\,.$$

Let us consider the set $\Lambda^* := \{0,2,\dots,2n-2\}\cup\{2n+1,2n+3,\dots,4n-1\}+(4n+2)\mathbb{Z}$, which contains $2n$ elements in each interval $\big[k(4n+2),(k+1)(4n+2)\big)$ of $4n+2$ numbers and hence has density $n/(2n+1)$. A direct calculation shows that $\Omega_N\cap(\Lambda^*-\Lambda^*)=\{0\}$, hence $L(N)\geq n/(2n+1)$. On the other hand we assert that for no Λ satisfying $\Omega_N\cap(\Lambda-\Lambda)=\{0\}$ can any interval $I=[k,k+2n]$ of $2n+1$ consecutive numbers contain more than n elements of Λ. Indeed, no pair of neighboring numbers belong to Λ, because $1\in\Omega_N$, and (at least) $n+1$ non-neighboring numbers can be placed into I only if all $m\in I$ with the same parity as k is contained. However, then both k and $k+2n$ is contained, having difference $2n\in\Omega_N$, a contradiction. Hence for a Λ satisfying our condition, the upper density can not exceed $n/(2n+1)$, which proves $L(N)=n/(2n+1)$.

Now we can compare the best estimate $\mathcal{T}_{\mathbb{Z}}(\Omega_N)\leq 1/L(N)=2+1/n$ arising from Theorem 3 to the exact value $2+1/\cos\frac{\pi}{2n+1}$ found above. It shows that application of Theorem 3 – although much better than the trivial estimate, but still – is not optimal in this case. This example highlights also the fact that number theoretical, intrinsic structural properties – like e.g. N being even or odd – essentially influence the values of the Turán constants and sharpness of the estimates we have.

Example 4. Another example of a nice set with nontrivial, but not sharp estimate arising from Theorem 3 is the unit disk D in $\mathbb{R}^2$ (with Lebesgue measure). The area of D is π and the right value of the Turán constant, first computed by Gorbachev [**Go01**], is $|D|/2^d=\pi/4$ in this case. Now D is the difference set of $H:=D/2$, and the best density we can have is, in fact, the *sphere packing constant* of $\mathbb{R}^2$. It is well-known [**AP95**] that the best packing is the regular hexagon lattice packing, hence $L(D)=2/\sqrt{3}$ and the arising estimate is $\sqrt{3}/2$. In comparison, note that the estimate of §3.5 gives $|D|/2=\pi/2$, while the estimate of Theorem 5 from the spectral approach does not apply, since the ball is *not* spectral. The above values compare as $\pi/4=0.785\dots<\sqrt{3}/2=0.866\dots<\pi/2=1.57\dots$.

Example 5. We see that for a general $\Omega\subset H-H$ or even $\Omega=H-H$ the "best translational set", (i.e. the maximal number of elements or the highest possible upper density), does not always achieve an exact bound of $\mathcal{T}_G(\Omega)$. In this respect it is worth mentioning that, on the other hand, results of Herz [**Hr56**], [**Hr60**] show that each subgroup Λ of G provides the theoretically best possible, sharp estimate for *some* open set Ω. E.g. if G is compact, and Λ is a finite subgroup having n elements, there exists a Borel set H with the properties $|H|=1/n$, $\Omega:=H-H$ is open, and $\Omega\cap\Lambda=\{0\}$. See also [**Ru62**, 7.4.1]. Clearly for this Ω and H we have that $H+\Lambda=G$ is a tiling, and $\mathcal{T}_G(\Omega)=1/n$, achieved by $\chi_H*\chi_{-H}$.

Example 6. The size of the Turán constant of a set Ω may be extremely small. Take for example in the group $G=\mathbb{Z}_{2n}$ the set $\Omega=\{0\}\cup K^c$, where K is the subgroup generated by 2. Let then $\Lambda=K$ and apply Theorem 2. It follows that $\mathcal{T}_G(\Omega)\leq 2$ while $|\Omega|=n+1$.

The same way we have $\mathcal{T}_{\mathbb{Z}}(\Omega)\leq 2$ for any subset $\Omega\subset(\{0\}\cup(2\mathbb{Z}+1))$ in view of Theorem 3 and considering the set $\Lambda:=2\mathbb{Z}$. (This covers the N odd case of Example 3, too.)

The generality of this example should be obvious.

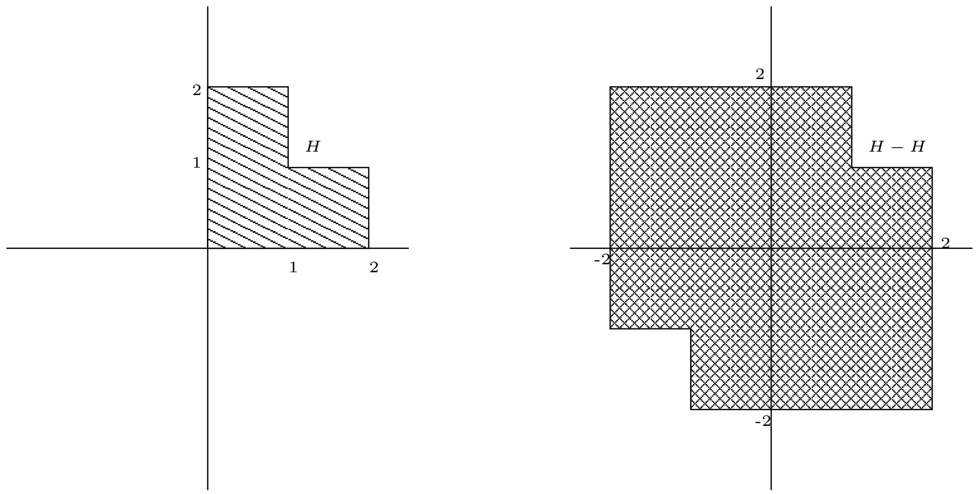

Figure 1: The Turán constant of $H - H$ (right) is equal to the area of H

3.4 The Turán constant of difference sets of tiles in $\mathbb{R}^d$ or $\mathbb{Z}^d$.

Here we show how to generalize the results in [**AB02**] (see also [**KR03**]). In [**AB02**] the Turán constant of convex polytopes which tile $\mathbb{R}^d$ by lattice translation was determined.

Actually being a polytope and *lattice* translation need not be assumed as it is a fact (see e.g. the references in [**KR03**]) that any convex body that tiles space by translation is a polytope and can also tile by lattice translation.

From Theorem 3 it follows that if H is any measurable set of finite measure that tiles $\mathbb{R}^d$ or $\mathbb{Z}^d$ by translation with Λ then the Turán constant of $H - H$ is equal to $1/\operatorname{dens}\Lambda = |H|$.

Whenever Ω is a convex body in $\mathbb{R}^d$ one can take $H = \frac{1}{2}\Omega$, so Theorem 3 is indeed a generalization of the result in [**AB02**].

However, Theorem 3 can determine the Turán constant of many more sets than those dealt with in [**AB02**], such as the one in Figure 1. The subset of the plane H shown on the left tiles the plane by translation hence its difference set shown on the right has Turán constant equal to $|H|$.

Example 7. Let $H \subset \mathbb{Z}^2$ be the three-element set $\{(0,0), (0,1), (1,0)\}$ and Ω be the difference set $H - H = \{(-1,0), (-1,1), (0,-1), (0,0), (0,1), (1,-1), (1,0)\}$. Then $|\Omega| = 7$, but H tiles $\mathbb{Z}^2$, hence Theorem 3 applies and yields $|H| = 3$. Observe that the set $\Lambda := \mathbb{Z}(1,1) + \mathbb{Z}(2,-1)$ provides a translational set. Indeed, any points $(n+2m, n-m)$ of Λ, and thus also of $\Lambda - \Lambda$, has the property that the first coordinate is congruent to the second $\mod 3$, hence $\Omega \cap \Lambda - \Lambda = \{(0,0)\}$. On the other hand all points of $\mathbb{Z}^2$ with the above congruence property belong to Λ, i.e. Λ is a subgroup of index 3. It follows that the density of Λ is $1/3$, and Theorem 3 gives the assertion.

3.5 The Turán constant of dispersed sets

As an application of Theorem 3 we show that, in $\mathbb{R}$, the Turán constant of a set of given length is the largest if the set is an interval. The construction extends to $\mathbb{Z}$, and even to $\mathbb{R}^d$ and $\mathbb{Z}^d$ giving a generally valid improvement of the trivial bound by about a factor of 2.

Theorem 6. *Let $\Omega \subseteq \mathbb{R}^d$ be an open set of finite measure m. Then we have*

$$\mathcal{T}_{\mathbb{R}^d}(\Omega) \leq \frac{m}{2} \ . \tag{29}$$

Let $\Omega \subseteq \mathbb{Z}^d$ be a set of size m containing the origin and denote by m^+ the number of lattice points in the "nonnegative half of Ω", i.e. in $\Omega \cap \left([0,\infty) \times \mathbb{Z}^{d-1}\right)$. Then we have

$$\mathcal{T}_{\mathbb{Z}^d}(\Omega) \leq m^+ \ . \tag{30}$$

Proof. Let us denote $P := [0,\infty) \times \mathbb{R}^{d-1}$ or $[0,\infty) \times \mathbb{Z}^{d-1}$, respectively, and put $\Omega^+ := \Omega \cap P$. Note that in $\mathbb{R}^d$ we simply have $m^+ := |\Omega^+| = m/2$. It is easy to see that Theorem 1 (on the equivalent formulations of the Turán constant), allows us to assume that Ω is bounded: so let $\Omega \subset B(0,r)$ with some fixed ball of radius r. Take a large parameter $L_0 > \max\{2,r\}$, define $L_k = L^{2^k} = L_{k-1}^2$ ($\forall k \in \mathbb{N}$), say, and put

$$Q_k := Q_{L_k}((L_k,0,\ldots,0)) = [0,2L_k] \times [-L_k,L_k]^{d-1} \quad (k \in \mathbb{N}), \quad Q_0 := \emptyset \ . \tag{31}$$

Note that $|Q_k| = (2L_k)^d$ in $\mathbb{R}^d$ and $(2L_k+1)^d$ in $\mathbb{Z}^d$. Define

$$S_k := Q_k \setminus (Q_{k-1} + \Omega) \quad (k \in \mathbb{N}) \ . \tag{32}$$

Obviously, S_k are closed sets of measure

$$|S_k| \geq |Q_k| - |Q_{k-1}+\Omega| \geq (2L_k)^d - ((2L_{k-1}+1)+2r)^d \geq 2^d L_k^d \left(1 - (\frac{2+r}{L_{k-1}})^d\right) \quad (k \in \mathbb{N}), \tag{33}$$

satisfying $(S_k - S_n) \cap \Omega = \emptyset$ for $k \neq n$. We aim at constructing the discrete set

$$\Lambda := \bigcup_{k=1}^{\infty} \Lambda_k, \qquad \Lambda_k \subset S_k \quad (k \in \mathbb{N}) \tag{34}$$

with as many as possible elements but satisfying $(\Lambda_k - \Lambda_k) \cap \Omega = \{0\}$. Note that if the latter condition is satisfied, then we will also have $(\Lambda - \Lambda) \cap \Omega = \{0\}$ in view of the respective property of $S_k \supset \Lambda_k$. So now we define the elements of Λ_k inductively by a "greedy algorithm" as follows. Let $\lambda_0^{(k)}$ be any element of the nonempty set S_k with first coordinate 0. Such an element clearly exists. Then for $n \geq 1$ take any

$$\begin{aligned} &\lambda_n^{(k)} := (x_{1,n},\ldots,x_{d,n}) \in \big(S_k \setminus \bigcup_{j=1}^{n-1} (\lambda_j^{(k)} + \Omega^+)\big) \\ &\quad \text{with} \\ &x_{1,n} = \min\left\{ x_1 \ : \ \exists\, x = (x_1,\ldots,x_d) \in \big(S_k \setminus \bigcup_{j=1}^{n-1} (\lambda_j^{(k)} + \Omega^+)\big) \right\} . \end{aligned} \tag{35}$$

Defining new elements $\lambda_n^{(k)}$ of Λ_k terminates in a finite number of steps, but not before $\cup_{j=1}^{n-1}(\lambda_j^{(k)} + \Omega^+)$ covers S_k, so with $m^+ := |\Omega^+|$ we must have

$$\#\Lambda_k \geq \frac{|S_k|}{|\Omega^+|} \geq \frac{2^d L_k^d \left(1 - (\frac{2+r}{L_{k-1}})^d\right)}{m^+} \quad (k \in \mathbb{N}) \ . \tag{36}$$

By construction, for any $n > j$ $\lambda_n^{(k)} - \lambda_j^{(k)} \in \Omega$ is not possible, hence $\Lambda - \Lambda \cap \Omega = \{0\}$. Moreover, in view of (36) we have

$$\overline{\operatorname{dens}}\, \Lambda \geq \limsup_{k\to\infty} \frac{\#\Lambda_k}{|Q_k|} \geq \limsup_{k\to\infty} \frac{\left(1 - (\frac{2+r}{L_{k-1}})^d\right)}{m^+} = \frac{1}{m^+} \ . \tag{37}$$

Now an application of Theorem 3 with Λ concludes the proof. □

Remark 4. For $d = 1$ (29) is sharp for intervals in $\mathbb{R}$. It is plausible, but we do not know if intervals are the only cases of equality.

Remark 5. As Ω is always symmetric, in $\mathbb{Z}$ we always have $m^+ = (m+1)/2$. The estimate (30) can also be sharp at least for $d = 1$. Take e.g. $\Omega = \Omega_0$ or Ω_1 from Example 3, or, more generally, take $\Omega := [-N,N]$. Then $m = 2N+1$, $m^+ = (m+1)/2 = N+1$, and the Fejér kernel shows that this value can be achieved. Thus $\mathcal{T}_{\mathbb{Z}}([-N,N]) = N+1$, and intervals have maximal Turán constants once again. However, here the sets $k[-N,N] := \{kn \ : \ |n| \leq N\}$ of similar size have equally large Turán constants, hence intervals are not the only extremal examples in $\mathbb{Z}$.

Remark 6. It can be proved that the asymptotic uniform upper density of all sets remain the same both in $\mathbb{R}^d$ and in $\mathbb{Z}^d$ if we define it replacing Q_R by RK with any other convex body K. Thus in the above proof one can consider the slightly modified basic sets $RQ(T)$, where $RQ(T)$ is the R-dilated copy of the unit box rotated by the isometry $T \in SO(d)$. If we choose T to be "irrational" in the sense that no lattice point (apart from the origin) moves to the hyperplane $\{x_1 = 0\}$, then with these sets a similar argument leads the same estimate but now with $m^+ = \#\Omega^+ = (m+1)/2$. We leave the details to the reader.

3.6 The Turán constant of an interval missing two points

Our next result shows the effect of forcing a positive definite function to vanish at a neighborhood of one point in an interval.

Theorem 7. *Suppose* $0 < b < a \le 2b$ *and let*

$$\Omega = (-a, -b) \cup (-b, b) \cup (b, a).$$

Then $\mathcal{T}_{\mathbb{R}}(\Omega) = \mathcal{T}_{\mathbb{R}}(-b, b) = b$.

Proof. Simply take $\Lambda = b\mathbb{Z}$ and apply Theorem 3 to obtain that $\mathcal{T}_{\mathbb{R}}(\Omega) \le b$. The other direction is obvious by the monotonicity of $\mathcal{T}_G(\cdot)$. $\square$

The condition $a < 2b$ is necessary in Theorem 7. Indeed, if $a > 2b$ then, with $c = \min\{b, (a-b)/2\} > b/2$ and $d := (a+b)/2$ the function $f := \chi_{(0,c)} * \chi_{(-c,0)} * (\delta_0 + \delta_d) * (\delta_0 + \delta_{-d})$, whose graph consists of three triangles centered at 0 and $\pm d$ of width $2c$ and heights 1 (for the central triangle) and 1/2 (for the other two) is positive definite and supported in Ω, yet has $f(0) = 2c$ and $\int_{\mathbb{R}} f = 4c^2$. Hence $\mathcal{T}_{\mathbb{R}}(\Omega) \ge 2c > b$.

§4. Upper bound from spectral sets

4.1 Proof of the bound from spectral sets

Proof of Theorem 4. Since T is a spectrum of H we have (see §1.3)

$$\begin{aligned} \operatorname{supp} \widehat{\chi_T} &\subseteq \{0\} \cup (H-H)^c \\ &\subseteq \{0\} \cup \Omega^c \\ &\subseteq \{0\} \cup \{f = 0\}. \end{aligned}$$

Hence $\widehat{f} + T = c\widehat{G}$ is a tiling and $c = |T| f(0)$, as $\int_{\widehat{G}} \widehat{f} = \left|\widehat{G}\right| f(0)$.

Since $\widehat{f} \ge 0$ in $\widehat{G}$ it follows that $\widehat{f}(0) \le c$ or

$$\sum_{x \in G} f(x) \le |T| f(0) = |H| f(0).$$

$\square$

4.2 Comparison of Theorems 2, 3 and 4, 5

First we give an example when Theorem 4 gives a better bound than any possible application of Theorem 2. Let $G = \mathbb{Z}_2^{12}$ and $H = \{e_1, e_2, \ldots, e_{12}\}$, where e_i is the vector in G with all zeros except at the i-th position where we have 1. The set H was recently shown by Tao [**Ta03**] to have a spectrum, and it is clear that H cannot tile G since $|H| = 12$ does not divide $|G| = 2^{12}$.

Let $\Omega = H - H$. This means that Ω consists of the all-zero vector plus all vectors in G with precisely two 1's, hence $|\Omega| = \binom{12}{2} + 1 = 67$.

By Theorem 4 we have that if $f : G \to \mathbb{C}$ is a positive definite function supported on Ω then

$$\sum_{x \in G} f(x) \le 12 f(0).$$

Suppose now that Theorem 2 applies with some $\Lambda \subseteq G$, such that $\Omega \cap (\Lambda - \Lambda) = \{0\}$. Since $\Omega = H - H$ this implies that $H + \Lambda \le G$ is a packing at level 1, hence $|\Lambda| \le \frac{1}{12}|G|$. In fact $|\Lambda| < \frac{1}{12}|G|$ as $|\Lambda|$ is an integer but $\frac{1}{12}|G|$ is not. Clearly then (7) is inferior than $\sum_{x \in G} f(x) \le 12 f(0)$ given by Theorem 4.

Tao [**Ta03**] also shows how to construct a domain (in fact, a finite union of unit cubes) in $\mathbb{R}^d$, $d \geq 5$, which is spectral but not a translational tile. Suppose H is such a domain. Theorem 5 shows that $\mathcal{T}_{\mathbb{R}^d}(H-H) \leq |H|$. We claim that Theorem 3 gives a worse upper bound for the set $\Omega = H - H$. Indeed, suppose that $\Lambda \subseteq \mathbb{R}^d$ is a set for which

$$\Omega \cap (\Lambda - \Lambda) = \{0\},$$

as required by Theorem 3, and that ρ is the upper density of Λ. Condition (38) means that $H+\Lambda$ is a packing, hence $|H| \mathrm{dens}\, \Lambda \leq 1$. The fact that H is not a tile implies (this requires a proof, an easy diagonal argument) that the inequality above is strict, so that $1/\rho > |H|$, which shows that any application of Theorem 3 gives a worse result than Theorem 5 for $H - H$.

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