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Structure of Tilings of the Line by a Function

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Abstract

A function $f \in L^1(\mathbb{R})$ tiles the line with a constant weight w using the discrete tile set A if $\sum_{a\in A} f(x-a) = w$ almost everywhere. A set A is of bounded density if there is a constant C such that $\#\{a \in A : \ n \le a \le n+1\} \le C$ for all integers n. This paper characterizes compactly supported $f \in L^1(\mathbb{R})$ that admit a tiling of $\mathbb{R}$ of bounded density. It shows that for such functions all tile sets of bounded density A are finite unions of complete arithmetic progressions. The results apply to some noncompactly supported $f \in L^1(\mathbb{R})$. The proofs depend on Cohen's theorem characterizing idempotent measures on locally compact abelian groups. We use a result of Meyer which, using Cohen's theorem, characterizes the collections of point masses on the real line whose Fourier transform is a locally finite measure with total variation that grows at most linearly.

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1. Introduction

Traditional problems of tiling concern whether or not a subset S of $\mathbb{R}^n$ can be tiled using a given set of allowed tile shapes ("prototiles"). Such problems may be formulated as that of representing the characteristic function χ_S as a sum of characteristic functions of sets isometric to prototiles using the allowed group of tile motions. A natural generalization is to relax this condition to allow tilings of χ_S using more general functions. The supports of the copies of the functions used in such a tiling may now overlap ("soft tiles"). Soft tilings are a special case of "soft packings", which have been studied to obtain bounds in sphere packing and coding theory [9]. Tilings by translates of functions arise naturally in wavelet theory: the scaling function for a compactly supported wavelet basis of $\mathbb{R}^n$ given by a multiresolution analysis must always have a lattice tiling of $\mathbb{R}^n$ in this generalized sense, see Strichartz [13], 1.17. Such tilings also arise in subdivision schemes in curve and surface design and in approximation, see [2, p. 14]. In addition, multiple tilings using copies of a set T are a special case of tilings by functions, in which the functions used are scaled characteristic functions $M^{-1}\chi_T$, where M is the multiplicity.

This paper studies tilings of the line $\mathbb{R}$ by translates of a single function $f \in L^1(\mathbb{R})$. A tile set A gives a *general tiling of (constant) weight* w provided that

$$\sum_{a \in A} f(x+a) = w \ , \tag{1.1}$$

for almost every (Lebesgue) $x \in \mathbb{R}$, where the convergence in (1.1) is absolute. The tile set A is required to be discrete, i.e. for each $T > 0$ the set $\{a \in A : |a| < T\}$ is finite, and we allow elements of A to occur with finite multiplicity. Our object is to determine which functions tile $\mathbb{R}$ and to specify the structure of possible tilings.

General tilings by a function f permit the pathology that mass can "leak out to infinity," cf. example 7.1 in §7. To exclude this pathology we restrict the class of allowed tilings. A tile set A is of *bounded density* if there is a constant $C > 0$ such that for all $T \in \mathbb{R}$,

$$\#\{a \in A : T \leq a < T+1\} \leq C .$$

The associated tiling (1.1) is called a *tiling of bounded density.* For nonnegative functions a general tiling is necessarily of bounded density, see Lemma 2.1.

This paper studies bounded density tilings and characterizes their possible structure for compactly supported functions. Lagarias and Wang [6] previously considered the special case of tilings of weight 1 by the characteristic function χ_T of a compact set T. They showed that all such tile sets are periodic, i.e. of the form $\alpha\mathbb{Z} + \{\beta_1, \ldots, \beta_J\}$, and that the differences $\beta_i - \beta_j$ of cosets are rational multiples of the period α. Here we show that for arbitrary functions a wider variety of tile sets occur.

Two tilings with weights w_1 and w_2, and tile sets A_1 and A_2, respectively, yield a combined tiling using the tile set $A_1 \cup A_2$, with weight $w_1 + w_2$. (We adopt the convention that the union of tile sets counts points with multiplicity.) Thus the set of all weights occurring from tilings of bounded density using a given function f form a semigroup $W(f)$, which we call the *weight semigroup of f*.

If a tile set A can be partitioned into two nonempty sets $A_1 \cup A_2$ such that each is separately a tile set for f, we say that the tile set A is *decomposable*; otherwise it is *indecomposable.*

To illustrate these notions, consider the trapezoidal function f pictured in Figure 1.1, which is given by the convolution of the characteristic functions $\chi_{[-1,1]}$ and $\chi_{[-\sqrt{2},\sqrt{2}]}$. It tiles $\mathbb{R}$ with the two different tile sets $A = 2\sqrt{2}\mathbb{Z}$ and $A = 2\mathbb{Z}$, having weights 2 and $6 - 2\sqrt{2} \cong 3.18$, respectively. Both of these tilings are indecomposable. To see this, note that if $A = A_1 \cup A_2$, then one of the A_i would have weight at most half that of A, which contradicts the fact that f has no tilings of weight less than 2, since f is nonnegative and $f(0) = 2$.

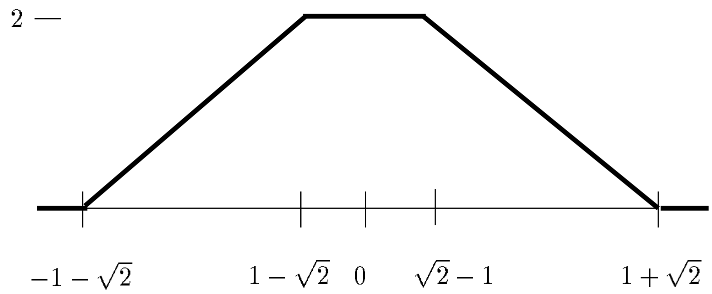

Figure 1.1: Trapezoidal Tile

Our main result gives a complete classification of the structure of the tiling sets possible for compactly supported functions.

Theorem 1.1. *Let $f \in L^1(\mathbb{R})$ have compact support, with $f \not\equiv 0$.*

(i) *Any tile set A of bounded density for f is a finite union of arithmetic progressions*

$$A = \bigcup_{j=1}^{J} (\alpha_j \mathbb{Z} + \beta_j) \ , \tag{1.2}$$

with all $\alpha_j > 0$.

(ii) *Every tile set of bounded density for f partitions into a finite union of indecomposable tile sets. Furthermore, each indecomposable tile set of bounded density A is periodic, i.e. there exists $\alpha > 0$ such that*

$$A = \bigcup_{j=1}^{J} (\alpha \mathbb{Z} + \beta_j) \ . \tag{1.3}$$

The expressions for tile sets A above allow $\beta_i = \beta_j$, to include tile sets with multiplicities. Note that (ii) implies (i), but our proof establishes (i) before (ii).

We complement Theorem 1.1 with a criterion for a function f to tile $\mathbb{R}$ with some tiling of bounded density.

Theorem 1.2. *A compactly supported function $f \in L^1(\mathbb{R})$ tiles $\mathbb{R}$ with some tile set A of bounded density if and only if the set of real zeros of its Fourier transform $\widehat{f}(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi} f(x)dx$ contains a set $\alpha\mathbb{Z}\backslash\{0\}$ for some $\alpha > 0$. If so then f tiles $\mathbb{R}$ with some weight w using the tile set $A' = (2\pi/\alpha)\mathbb{Z}$.*

Using this result we establish the following converse to part (i) of Theorem 1.1.

Theorem 1.3. *If A consists of a finite union of complete arithmetic progressions*

$$A = \bigcup_{j=1}^{J} (\alpha_j \mathbb{Z} + \beta_j) \ , \ \textit{all } \alpha_j > 0,$$

then there exists a compactly supported function $f \in L^1(\mathbb{R})$ which tiles $\mathbb{R}$ using the tile set A, with weight $w > 0$.

The corresponding converse question to (ii) of Theorem 1.1, concerning which sets of the form (1.3) occur as the tile set of some indecomposable tiling, we leave unresolved. We give in §7 an example of an indecomposable tile set (1.3) in which the differences $\beta_i - \beta_j$ are not all rational multiples of the period α. This shows that the rationality result of [6], valid for weight 1 tilings for characteristic functions χ_S of compact sets S, is not generally true for compactly supported functions $f \in L^1(\mathbb{R})$.

Theorems 1.1–1.3 are proved in §6. Theorem 1.1 is deduced as a consequence of a stronger result (Theorem 5.1) that applies also to some functions not of compact support. The proof of Theorem 5.1 relies heavily on Cohen's classification of idempotent measures on arbitrary locally compact abelian groups [3], [11, p. 59], i.e. measures whose Fourier transform takes only the values 0 and 1. The group on which we use Cohen's theorem is the Bohr compactification of the real line $\overline{\mathbb{R}}$. We use a slight modification of a result of Meyer [8] on the structure of infinite collections of point masses whose Fourier transform is a measure having total variation growing at most linearly (Theorem 4.2).

The tiling structure of Theorem 1.1 does not generalize to nonnegative bounded functions $f \in L^\infty(\mathbb{R})$ having infinite mass. There are examples of (infinite) aperiodic sets $A, B \subseteq \mathbb{Z}$ such that every element $m \in \mathbb{Z}$ has a unique representation $m = a + b$, with $a \in A$ and $b \in B$. One such example takes $A = \{a : \ a \geq 0 \text{ is a sum of distinct odd powers of } 2\}$, see Tijdeman [14], §4. If $T = [0,1] + B$ then $f = \chi_T$ tiles $\mathbb{R}$ with the aperiodic set A of bounded density.

The contents of the paper are as follows. In §2 we derive some elementary results related to tilings of bounded density. For a nonnegative $f \in L^1(\mathbb{R})$ every general tiling is a tiling of bounded density. For $f \in L^1(\mathbb{R})$ if $\int_{-\infty}^{\infty} f(x)\, dx = 0$ then all tilings of bounded density have weight 0, while if $\int_{-\infty}^{\infty} f(x)\, dx > 0$ such tilings have nonnegative weights. If f is compactly supported with $L = \int_{-\infty}^{\infty} f(x)\, dx > 0$ the minimal weight of a tiling of bounded density is at least $L/(2R)$, where f is supported in $[-R, R]$.

In §3 we derive for a class of $f \in L^1(\mathbb{R})$ conditions on the Fourier transform of f which are necessary to have a tiling of $\mathbb{R}$ of bounded density. We also derive a sufficient condition.

In §4 we state results concerning Cohen's idempotent theorem, and prove a variant of a result of Meyer [8] giving sufficient conditions for a measure $\mu_A = \sum_{a \in A} \delta_a$ to have A be a finite union of arithmetic progressions. (Here δ_a is a point mass at $a \in \mathbb{R}$.)

In §5 we prove a structure theorem for tile sets A of bounded density for a class of functions $f \in L^1(\mathbb{R})$ whose Fourier transform $\hat{f} \in C^\infty(\mathbb{R})$ has a discrete zero set which contains $O(R)$ zeros in $[-R, R]$ as $R \to \infty$. This class of functions includes all compactly supported $f \in L^1(\mathbb{R})$. In §6 we deduce Theorems 1.1, 1.2 and 1.3, using this result.

In §7 we present a collection of examples of compactly supported functions in $L^1(\mathbb{R})$ showing various senses in which results cannot be improved. These include functions that have tilings that are not of bounded density, a function having uncountably many indecomposable tilings all having the same weight, a function having a weight semigroup $W(f)$ requiring an infinite number of generators, and a function which has an indecomposable tiling with rational period but with two cosets having an irrational difference.

Notation. The symbol $f(x) \ll g(x)$ means that there exists a positive constant C such that $f(x) \le Cg(x)$ for the indicated range of x.

The characteristic function $\chi_S(x)$ of a set S is 1 if $x \in S$ and is 0 otherwise.

We denote by $|f|_E$ the supremum of the function $|f|$ on the set E.

We define the counting function of a set $A \subseteq \mathbb{R}$ by:

$$N_A(T_1, T_2) = \#(A \cap [T_1, T_2]).$$

2. Tilings of Bounded Density

The assumption of bounded density allows information about tilings to be extracted using arguments that estimate areas covered by the "tiles" f.

Lemma 2.1. *If $f \in L^1(\mathbb{R})$ is a nonnegative function with $\int_{-\infty}^{\infty} f(x)\, dx > 0$ then every general tiling of $\mathbb{R}$ by f is a tiling of bounded density.*

Proof. By hypothesis

$$\sum_{a \in A} f(x-a) = w \quad \text{almost everywhere },$$

and $w > 0$ since $\int_{-\infty}^{\infty} f(x)dx > 0$ rules out the zero function. Choose $R > 1$ so that $J = \int_{-R}^{R} f(x)\, dx > 0$. Now

$$\begin{aligned}
(2R+1)w &= \int_{T-R}^{T+R+1} \sum_{a \in A} f(x-a)\, dx \\
&\geq \int_{T-R}^{T+R+1} \sum_{T<a<T+1} f(x-a)\, dx \\
&\geq N_A(T, T+1) \int_{-R}^{R} f(x)\, dx .
\end{aligned}$$

Thus $N_A(T, T+1) \leq (2R+1)w/J$ is bounded independent of T. □

The usefulness of the bounded density assumption lies in the following simple fact.

Lemma 2.2. *If $f \in L^1(\mathbb{R})$ and A is any set of bounded density then*

$$G(x) = \sum_{a \in A} f(x-a)$$

is absolutely convergent almost everywhere and is locally integrable.

Proof. Suppose that $N_A(T, T+1) \leq C$ for all T. Then

$$\begin{aligned}
\int_{T}^{T+1} \left| \sum_{a \in A} f(x-a) \right| dx &\leq \int_{T}^{T+1} \sum_{a \in A} |f(x-a)|\, dx \\
&\leq \sum_{a \in A} \int_{T}^{T+1} (|f(x - \lfloor a \rfloor)| + |f(x - \lfloor a \rfloor - 1)|)\, dx \\
&\leq 2C \sum_{n \in \mathbb{Z}} \int_{T}^{T+1} |f(x-n)|\, dx \\
&= 2C \int_{-\infty}^{\infty} |f(x)|\, dx < \infty .
\end{aligned}$$

This gives local integrability, and implies that $G(x)$ is defined as an absolutely convergent series almost everywhere. □

Lemma 2.3. *Suppose that $f \in L^1(\mathbb{R})$ and set $L = \int_{-\infty}^{\infty} f(x)\, dx$.*

(i) *If* $L = 0$ *then any tiling of* $\mathbb{R}$ *of bounded density by* f *has weight 0.*

(ii) *If* $L > 0$ *then any tiling of bounded density by* f *has weight* $w \geq 0$*. The tiling set* A *has asymptotic density* w/L*.*

Proof. We first derive a formula to estimate

$$N_A(T) = N_A(-T, T) = \#\{a \in A : \ -T \leq a \leq T\} .$$

The assumption of bounded density gives

$$N_A(T+U) - N_A(T) \leq 2CU, \text{ for } U \geq 1 , \tag{2.1}$$

and in particular

$$N_A(T) \leq 2CT \quad \text{for } T \geq 1 . \tag{2.2}$$

Since A is a tile set,

$$\sum_{a \in A} f(x-a) = w \quad \text{almost everywhere} .$$

We integrate this relation over the interval $[-T, T]$, to obtain

$$\begin{aligned} 2wT &= \textstyle\int_{-T}^{T} \sum_{a \in A} f(x-a) \, dx \\ &= N_A(T-R)L + E(T) , \end{aligned} \tag{2.3}$$

in which $E(T)$ is a remainder term, given by a sum of three terms:

$$\begin{aligned} E_1(T) &= -\int_{|x|>T} \sum_{\substack{a \in A \\ |a| \leq T-R}} f(x-a) \, dx , \\ E_2(T) &= \int_{-T}^{T} \sum_{\substack{a \in A \\ T-R \leq |a| \leq T+R}} f(x-a) \, dx , \end{aligned}$$

and

$$E_3(T) = \int_{-T}^{T} \sum_{\substack{a \in A \\ |a| > T+R}} f(x-a) \, dx .$$

To bound the $E_i(T)$, set $\int_{-\infty}^{\infty} |f(x)| dx = M$ and, given $\epsilon > 0$, pick R so large that $\int_{|x|>R} |f(x)| dx < \epsilon$. Then, assuming $T > R > 1$ and using (2.1), we obtain

$$\begin{aligned} |E_1(T)| &\leq N_A(T-R) \int_{|x|>R} |f(x)| \, dx \\ &\leq \epsilon N_A(T-R) \leq 2C\epsilon T \\ |E_2(T)| &\leq (N_A(T+R) - N_A(T-R)) \int_{-\infty}^{\infty} |f(x)| dx \\ &\leq 2CRM . \end{aligned}$$

To bound $E_3(T)$, we use the bounded density of A, to obtain

$$|E_3(T)| \le 2C \int_{-T}^{T} \sum_{\substack{n\in\mathbb{Z} \\ |n|>T+R}} |f(x-n)| \, dx \ .$$

Now split the interval $[-T,T]$ into $2T$ intervals of length 1 and rearrange terms to obtain

$$|E_3(T)| \le 4CT \int_{|x|>R} |f(x)| \, dx \le 4C\epsilon T \ .$$

Substituting these estimates into (2.3), and using (2.1) to replace $N(T-R)$ with $N(T)$ yields

$$|N_A(T)L - 2wT| \le 6C\epsilon T + 2CR(L+M) \ . \tag{2.4}$$

This inequality relates w and $N_A(T)$ for all L.

(i) Suppose that $L = 0$. Then (2.4) yields

$$|w| \le 6C\epsilon + \frac{CRM}{T} \ .$$

Letting $T \to \infty$, and then letting $\epsilon \to 0$, gives $w = 0$.

(ii) Suppose that $L > 0$. Then (2.4) can be rewritten

$$\left| \frac{N_A(T)}{2T} - \frac{w}{L} \right| \le \frac{3C}{L}\epsilon + \frac{CR(L+M)}{LT} \ .$$

This yields

$$\frac{3C}{L}\epsilon \ge \limsup_{T\to\infty} \left(\frac{N_A(T)}{2T} - \frac{w}{L} \right) \ge \liminf_{T\to\infty} \left(\frac{N_A(T)}{2T} - \frac{w}{L} \right) \ge -\frac{3C}{L}\epsilon \ .$$

Now let $\epsilon \to 0$ to obtain

$$\lim_{T\to\infty} \frac{N_A(T)}{2T} = \frac{w}{L} \ ,$$

which shows A has asymptotic density w/L. Since $N_A(T)$ is nonnegative, we have $w \ge 0$, proving (ii). □

It seems that the condition $L = \int_{-\infty}^{\infty} f(x) \, dx > 0$ should rule out the existence of tilings of bounded density with weight $w = 0$. However we shall only demonstrate this for functions of compact support. For those functions the following lemma bounds $\inf W(f)$ away from zero.

Lemma 2.4. *Suppose that* $f \in L^1(\mathbb{R})$ *with* $L = \int_{-\infty}^{\infty} f(x)\, dx > 0$, *and that* $\operatorname{supp} f \subseteq [-R, R]$. *If* f *has a tiling of* $\mathbb{R}$ *of bounded density and of weight* w *then*

$$w \geq \frac{L}{2R} \ . \tag{2.5}$$

Proof. We know that $w \geq 0$ by Lemma 2.2, and first show that $w \neq 0$. If $w = 0$ then Lemma 2.2 shows that the tile set A has density zero. In particular there exists a point $a_0 \in A$, such that the open interval $(a_0, a_0 + 3R)$ contains no point of A. Then the rightmost Lebesgue point x^* of $f(x - a_0)$ has a neighborhood outside the support of any other translate $f(x - a)$, $a \in A \backslash \{a_0\}$. Then $\sum_{a \in A} f(x - a) = mf(x - a_0) \not\equiv 0$ in a small enough neighborhood of x^*, which contradicts $w = 0$ (m is the multiplicity of a_0 in A).

Now suppose $w > 0$. Lemma 2.2 asserts that A has density w/L. If $w/L < 1/(2R)$ then for any ϵ with $0 < \epsilon < 2R - L/w$ we can find an open interval $(a, a + 2R + \epsilon)$ which contains no point of A. Then the interval $(a + R, a + R + \epsilon)$ is disjoint from the support of all $f(x - a)$, $a \in A$, hence

$$\sum_{a \in A} f(x - a) = 0 \quad \text{for } x \in (a + R, a + R + \epsilon) \ .$$

This contradicts $w \neq 0$. It follows that $w/L \geq 1/(2R)$. □

3. A Spectral Condition for Tiling

Let $f \in L^1(\mathbb{R})$. We derive a necessary condition on the Fourier transform $\widehat{f}(\xi)$ for f to tile the line. We also derive a sufficient condition.

For conventions concerning Fourier transforms, measures, and distributions we generally follow Rudin [12]. The Fourier transform of $f \in L^1(\mathbb{R})$ and its inversion are normalized as

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} f(x) dx; \qquad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} \widehat{f}(\xi) d\xi \ .$$

With this definition $\widehat{\widehat{f}}(x) = 2\pi f(-x)$.

A *measure* is a (complex-valued) regular Borel measure on $\mathbb{R}$, which may have infinite mass. A *Radon measure* μ is a measure on $\mathbb{R}$ which is locally *finite*, i.e., it is finite on compact subsets of $\mathbb{R}$. The measure consisting of a point mass at $a \in \mathbb{R}$ is denoted δ_a. We study the measure μ_A associated to a discrete set A by

$$\mu_A = \sum_{a \in A} \delta_a \ , \tag{3.1}$$

which is a Radon measure. A Radon measure μ is *uniformly locally finite* if there is a constant C such that

$$|\mu|(T, T+1) \leq C, \quad \text{all } T \in \mathbb{R} \ .$$

The measure μ_A of (3.1) is uniformly locally finite if and only if A is of bounded density.

In order to define the Fourier transform we work in the space $\mathcal{S}'$ of tempered distributions. Let $\mathcal{D}$ denote the space of compactly supported C^∞-functions (called *test functions*). The distributions $\mathcal{D}'$ are the continuous linear functions on $\mathcal{D}$ with respect to its Frechét topology. The action of the distribution β on the test function g is denoted by $\langle g, \beta \rangle$. The *Schwartz class* $\mathcal{S}$ consist of C^∞-functions g such that $\left\| |x|^\alpha g^{(m)}(x) \right\|_\infty < \infty$ for all $\alpha > 0$ and all nonnegative integers $m \geq 0$. The Fourier transform is a continuous linear bijection from $\mathcal{S}$ to itself. The *tempered distributions* $\mathcal{S}'$ are the continuous linear functionals on $\mathcal{S}$ with its Frechét topology. We identify $\mathcal{S}'$ with a subset of $\mathcal{D}'$. The Fourier transform is a continuous linear mapping from $\mathcal{S}'$ to $\mathcal{S}'$ whose inverse is also continuous, with $\widehat{\beta}$ defined by

$$\langle g, \widehat{\beta} \rangle = \langle \widehat{g}, \widetilde{\beta} \rangle, \quad g \in \mathcal{S}, \beta \in \mathcal{S}',$$

where $\widetilde{\beta}(x) = \overline{\beta(-x)}$. For each $\phi \in \mathcal{S}$ and $\beta \in \mathcal{S}'$ we define $\phi\beta \in \mathcal{S}'$ by $\langle g, \phi\beta \rangle = \langle \overline{\phi} g, \beta \rangle$, for all $g \in \mathcal{S}$. We define the convolution $g \star \beta$ of any $g \in \mathcal{S}$ with any $\beta \in \mathcal{S}'$ by defining its Fourier transform: $\widehat{g \star \beta} = \widehat{g} \cdot \widehat{\beta}$.

The *support* of $\beta \in \mathcal{S}'$, denoted $\operatorname{supp} \beta$, is defined as the smallest closed set $X \subseteq \mathbb{R}$ for which $\phi \in C_c^\infty(\mathbb{R} \backslash X)$ implies $\langle \phi, \beta \rangle = 0$.

We identify locally integrable functions and Radon measures with distributions, as in [12, 6.11]. The distribution associated with the point measure δ_a is sometimes called the *Dirac delta function* centered at a. The Radon measure μ_A is a distribution, but it need not be a tempered distribution. A sufficient condition for μ_A to be a tempered distribution is the growth condition

$$\#\{a \in A : \ |a| \leq R\} \ll R^c, \quad \text{as } R \to \infty \ , \tag{3.2}$$

for some $c > 0$.

Suppose now that A is a set of bounded density. Then (3.2) holds, hence the measure μ_A is a tempered distribution, with Fourier transform $\widehat{\mu_A} \in \mathcal{S}'$. For $A = \alpha\mathbb{Z}$ the *Poisson summation formula* asserts that

$$\widehat{\mu_A} = \left(\sum_{n\in\mathbb{Z}} \delta_{n\alpha} \right)^{\wedge} = \frac{2\pi}{|\alpha|} \left(\sum_{n\in\mathbb{Z}} \delta_{2\pi n/\alpha} \right) . \tag{3.3}$$

In this case $\widehat{\mu_A}$ is a Radon measure; in general $\widehat{\mu_A}$ need not be a measure.

Harmonic analysis comes into the study of translational tilings via the following theorem.

Theorem 3.1. *Let $f \in L^1(\mathbb{R})$ have a Fourier transform $\widehat{f} \in C^\infty(\mathbb{R})$ and let $A \subseteq \mathbb{R}$ be a set of bounded density.*

(i) *If f tiles $\mathbb{R}$ with weight w using the tile set A then*

$$\mathrm{supp}\, \widehat{\mu_A} \subseteq B = \{0\} \cup \{\xi \in \mathbb{R} : \ \widehat{f}(\xi) = 0\} .$$

(ii) *If $\widehat{\mu_A}$ is a Radon measure and if* $\mathrm{supp}\, \widehat{\mu_A} \subseteq B$ *then A is a tile set of $\mathbb{R}$ using f, with some weight w.*

Remark. The idea of the proof is as follows. The fact that f tiles with A can be written

$$f \star \mu_A = w .$$

Taking the Fourier transform gives

$$\widehat{f} \cdot \widehat{\mu_A} = (2\pi w)\delta_0 .$$

This in turn implies that $\widehat{f}$ vanishes on the support of $\widehat{\mu_A}$ except at 0. However neither $f \star \mu_A$ nor $\widehat{f} \cdot \widehat{\mu_A}$ is well-defined in general, hence the introduction of the function $F \in \mathcal{S}$ in the proof below.

Proof. (i) Note that B is a closed set, $\widehat{f}$ being a continuous function. To prove that $\mathrm{supp}\, \widehat{\mu_A} \subseteq B$ we need to show that $\langle \phi, \widehat{\mu} \rangle = 0$ for every $\phi \in C_c^\infty(\mathbb{R}\backslash B)$. Fix such a ϕ and take $\psi \in \mathcal{S}$ to have $\widehat{\psi} \in C_c^\infty(\mathbb{R})$ with $\widehat{\psi}(0) = 1$ and $\widehat{\psi}(\xi) \neq 0$, for all ξ in an interval that contains the support of ϕ. Write $F = \psi \star f$. We first prove that $F \star \mu_A = w$. Indeed $(F \star \mu_A)(x) = \sum_{a\in A} F(x-a)$, with the sum converging absolutely since $F \in \mathcal{S}$. Next

$$\sum_{a\in A} F(x-a) \ = \ \sum_{a\in A} \int_{-\infty}^{\infty} f(x-a-t)\psi(t)\, dt$$

$$= \int_{-\infty}^{\infty} \psi(t) \sum_{a\in A} f(x-t-a)\ dt$$

$$= w \int_{-\infty}^{\infty} \psi(t) dt = w\ .$$

The interchange of summation and integration is justified using Fubini's theorem. For this we need that $\sum_{a\in A} \int_{-\infty}^{\infty} |\psi(t)| \cdot |f(x-t-a)|\ dt < \infty$, which is true because the set A is of bounded density and $|\psi(t)|$ decreases faster than any power of t.

Since $\widehat{\psi}$ has compact support and $\widehat{f}$ is smooth the function $\widehat{\psi}\widehat{f}$ is in $\mathcal{S}$ and so is then $F = \psi \star f$. Taking the Fourier transform of $F \star \mu_A = w$ we get

$$\widehat{F}\,\widehat{\mu_A} = (2\pi w)\delta_0\ .$$

Notice that $\widehat{F}$ and $\widehat{f}$ have the same zero set within an open interval that contains the support of ϕ. It follows that $\phi/\overline{\widehat{F}}$ is a smooth function whose compact support is disjoint from B. We get

$$\langle \phi, \widehat{\mu_A} \rangle = \langle \phi/\overline{\widehat{F}}, \widehat{F}\widehat{\mu_A} \rangle = \langle \phi/\overline{\widehat{F}}, (2\pi w)\delta_0 \rangle = 0\ ,$$

since 0 is not in the support of ϕ. This proves (i).

(ii) Let $\psi \in \mathcal{S}$ be arbitrary and define $F = \psi \star f$. Consider now $\phi \in C_c^\infty(0, +\infty)$. We have

$$\langle \phi, \widehat{F}\widehat{\mu_A} \rangle = \langle \phi\overline{\widehat{F}}, \widehat{\mu_A} \rangle\ . \tag{3.4}$$

But $\widehat{F}$ vanishes wherever $\widehat{f}$ does. That is, $\widehat{F}$ vanishes throughout the support of $\widehat{\mu_A}$ (with the possible exception of 0) and since $\widehat{\mu_A}$ is a locally finite measure the second term in (3.4) is 0 (this is not necessarily true without the assumption the $\widehat{\mu_A}$ is a locally finite measure). Since ϕ was arbitrary in $C_c^\infty(0, +\infty)$, $\widehat{F}\widehat{\mu_A}$ has no support in $(0, \infty)$. Using a similar argument for $\phi \in C_c^\infty(-\infty, 0)$, we conclude that the support of the measure $\widehat{F}\widehat{\mu_A}$ is $\{0\}$. Therefore

$$\widehat{F}\widehat{\mu_A} = \left(\widehat{F}(0)\widehat{\mu_A}(\{0\})\right)\delta_0 = \left(\int_{-\infty}^{\infty} \psi(t)\ dt\right) \cdot \left(\widehat{f}(0)\widehat{\mu_A}(\{0\})\right)\delta_0\ ,$$

which implies

$$F \star \mu_A = (2\pi)^{-1}\widehat{f}(0)\widehat{\mu_A}(\{0\}) \int_{-\infty}^{\infty} \psi(t)\ dt = w \int_{-\infty}^{\infty} \psi(t)\ dt\ ,$$

for $w = (2\pi)^{-1}\widehat{f}(0)\widehat{\mu_A}(\{0\})$.

By Lemma 2.2 the function $G(x) = \sum_{a\in A} f(x-a)$ is locally integrable and we need to prove $G(x) = w$ almost everywhere. For *all* $x \in \mathbb{R}$ we have (there is no exceptional set of measure zero since $F \in \mathcal{S}$)

$$\begin{aligned} w\int_{-\infty}^{\infty} \psi(t)\, dt &= \sum_{a\in A} F(x-a) \\ &= \int_{-\infty}^{\infty} \psi(t)\left(\sum_{a\in A} f(x-t-a)\right) dt\ , \end{aligned}$$

with the interchange of summation and integration justified as in part (i). Since $\psi \in \mathcal{S}$ is arbitrary the function (of t) $G(x-t) = \sum_{a\in A} f(x-a-t)$ is equal to the constant function w as a tempered distribution. Being a locally integrable function $G(x-t)$ is equal to w a.e.(t), therefore $\sum_{a\in A} f(x-a) = w$ for almost all $x \in \mathbb{R}$. $\square$

4. Idempotent Measures and a Theorem of Meyer

To state Cohen's theorem on idempotent measures we first need the following definition. (For definitions of the dual group and the Fourier transform for locally compact abelian groups see, for example, [11].)

Definition 4.1. The *ring of cosets* of a locally compact abelian group Γ is the smallest set which contains all open cosets of Γ and which is closed under finite unions, finite intersections and complements.

As an example, when $\Gamma = \mathbb{Z}$ every subgroup is of the form $m\mathbb{Z}$ for a fixed $m \geq 0$. All these groups are open and so every coset is of the form $x + m\mathbb{Z}$ (when $m = 0$ the coset consists of one point). The ring of cosets of $\mathbb{Z}$ consists thus of all sets which are *eventually* periodic, that is they are finite unions of doubly infinite arithmetic progressions up to the addition or removal of a finite set of integers.

Cohen's Idempotent Theorem (Cohen [3], [11, p. 59]) *Let G be a locally compact abelian group and Γ its dual group. If μ is a finite Borel measure on G and is such that $\widehat{\mu}(\gamma) \in \{0,1\}$ for all $\gamma \in \Gamma$ then the support of $\widehat{\mu}$, which is*

$$\operatorname{supp}\widehat{\mu} = \{\gamma \in \Gamma :\ \widehat{\mu}(\gamma) = 1\}\ , \tag{4.1}$$

is in the ring of cosets of Γ.

(Such measures are called *idempotent*, the reason being that $\mu \star \mu = \mu$.)

The next theorem follows easily from Cohen's Idempotent Theorem, using an argument of Cohen [3, p. 204].

Theorem 4.1. *Let G be a locally compact abelian group and Γ its dual group. If μ is a finite Borel measure on G and the range of $\widehat{\mu}$ is a finite set $S = \{s_1, \ldots, s_n\} \subseteq \mathbb{C}$ then, for each $j = 1, \ldots, n$, the preimage of s_j*

$$(\widehat{\mu})^{-1}(s_j) = \{\gamma \in \Gamma : \ \widehat{\mu}(\gamma) = s_j\}$$

is in the ring of cosets of Γ.

Proof. Fix $j \in \{1, \ldots, n\}$. Let $P(z) = a_m z^m + \cdots + a_0$ be a polynomial that takes the value 1 on s_j and 0 on s_k for $k \neq j$. Define the measure ν as the convolution polynomial

$$\nu = P(\mu) = a_m \mu^{\star m} + \cdots + a_1 \mu + a_0,$$

where $\mu^{\star k} = \mu \star \cdots \star \mu$ (k times), which is well-defined since μ is a finite measure. Then, for each $\gamma \in \Gamma$, we have

$$\widehat{\nu}(\gamma) = P\left(\widehat{\mu}(\gamma)\right),$$

which implies

$$(\widehat{\mu})^{-1}(s_j) = \{\gamma \in \Gamma : \ \widehat{\mu}(\gamma) = s_j\} = \{\gamma \in \Gamma : \ \widehat{\nu}(\gamma) = 1\}.$$

Now ν is an idempotent measure on G, so by Cohen's Idempotent Theorem we have that $(\widehat{\mu})^{-1}(s_j)$ is in the ring of cosets of Γ. □

Take for example the case of $G = \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ and, consequently, $\Gamma = \mathbb{Z}$. As we saw, the ring of cosets of $\mathbb{Z}$ consists of all eventually periodic sets and it is precisely those sets that support the Fourier transform of an idempotent measure on $\mathbb{T}$. Of course, the converse of the theorem is also true and is easy to prove: for every A in the coset ring of Γ, the characteristic function $\chi_A(\gamma)$ is the Fourier transform of a finite Radon measure on G.

Theorem 4.1 allows us to prove the following variation of a result of Meyer [8, p. 25].

Theorem 4.2. *Let $A \subseteq \mathbb{R}$ be a discrete set and μ be the Radon measure*

$$\mu = \sum_{a \in A} c_a \delta_a, \quad c_a \in S,$$

where $S \subseteq \mathbb{C}\setminus\{0\}$ is a finite set. Suppose that μ is a tempered distribution, and that $\widehat{\mu}$ is a Radon measure on $\mathbb{R}$ which satisfies

$$|\widehat{\mu}|([-R,R]) \ll R, \text{ as } R \to \infty \ . \tag{4.2}$$

Then the set A is of the form

$$A = F \Delta \bigcup_{j=1}^{k} (\alpha_j \mathbb{Z} + \beta_j) \ , \tag{4.3}$$

where $\alpha_j > 0, \beta_j \in \mathbb{R}$, and the set $F \subseteq \mathbb{R}$ is finite (here Δ denotes symmetric difference of sets).

In Meyer's original formulation in place of condition (4.2) there was the stronger requirement that $\widehat{\mu_A}$ was uniformly locally finite, i.e. $|\widehat{\mu_A}|([x, x+1]) \le C$, for all $x \in \mathbb{R}$. We do need the relaxed hypothesis (4.2) for our application. The proof of Theorem 4.2 is along the same lines as that given by Meyer. A result of Córdoba [4] establishes the same structure (4.3) without using Cohen's theorem but under the extra assumption that $\widehat{\mu}$ is a linear combination of point masses with *nonnegative* coefficients (no growth condition like (4.2) is then needed).

Proof. Let $\phi \in C_c^\infty(-1,1), \phi(0) = 1$, so that its Fourier transform satisfies $|\widehat{\phi}(\xi)| \ll |\xi|^{-\alpha}$ for all $\alpha > 0$. For positive integers n define the functions

$$\mu_n(x) = \phi(nx) \star \mu(x) \ .$$

Their Fourier transforms satisfy

$$\widehat{\mu}_n(\xi) = \frac{1}{n} \widehat{\phi}(\xi/n) \, \widehat{\mu}(\xi) \ ,$$

(as distributions) hence the $\widehat{\mu}_n$ are all measures. We claim that the measures $\widehat{\mu_n}$ are uniformly bounded measures, i.e. $|\widehat{\mu_n}|(\mathbb{R}) \ll 1$. Indeed

$$|\widehat{\mu_n}|([-n,n]) \le \frac{1}{n} \, ||\widehat{\phi}||_\infty \, |\widehat{\mu}|([-n,n]) \ll 1 \ ,$$

by our assumption on the growth of $|\widehat{\mu}|([-n,n])$. Furthermore, if $2^k \gg n$ we have (using the fact that $|\widehat{\phi}(\xi)| \ll |\xi|^{-2}$ as $\xi \to \infty$)

$$|\widehat{\mu_n}|([2^k, 2^{k+1}]) \ \ll \ \frac{1}{n} \, |\widehat{\phi}|_{[2^k/n, 2^{k+1}/n]} \, |\widehat{\mu}|([0, 2^{k+1}])$$

$$\ll \frac{1}{n}\left(\frac{2^k}{n}\right)^{-2} 2^k \ll \frac{n}{2^k} \ ,$$

which implies

$$|\widehat{\mu_n}|([n,\infty)) \ll n \sum_{2^{k+1}\geq n} \frac{1}{2^k} \ll 1 \ ,$$

and with the similar inequality for $|\widehat{\mu_n}|((-\infty,-n])$ the proof of the claim is complete.

Notice also that $\lim_{n\to\infty}\mu_n(x) = c_x$ if $x \in A$ and is 0 otherwise. This is a consequence of the fact that A is discrete and the support of $\phi(nx)$ shrinks to 0.

We use the following properties of $\overline{\mathbb{R}}$, the Bohr compactification of $\mathbb{R}$ (see for example [11, p. 30]).

1. $\overline{\mathbb{R}}$ is the dual group of $\mathbb{R}_d$, the real line with the discrete topology. Therefore $\overline{\mathbb{R}}$ is a compact group.

2. $\mathbb{R} \subseteq \overline{\mathbb{R}}$ as topological spaces and $\mathbb{R}$ is dense in $\overline{\mathbb{R}}$. Identifying the continuous functions on $\overline{\mathbb{R}}$ with bounded continuous functions on $\mathbb{R}$ we get that

$$C(\overline{\mathbb{R}}) \subseteq C(\mathbb{R}) \cap L^\infty(\mathbb{R})$$

is a Banach space inclusion.

Since the measures $\widehat{\mu_n}$ are uniformly bounded they act on all bounded continuous functions of $\mathbb{R}$, and consequently also on all continuous functions on $\overline{\mathbb{R}}$, that is they constitute a uniformly bounded family of linear functionals on $C(\overline{\mathbb{R}})$. By the Banach-Alaoglu theorem there exists a measure ν on $\overline{\mathbb{R}}$ such that for every $f \in C(\overline{\mathbb{R}})$ there is a subsequence of $\widehat{\mu_n}$, call it again $\widehat{\mu_n}$, such that

$$\langle f, \widehat{\mu_n}\rangle \to \langle f, \nu\rangle, \quad \text{as } n \to \infty.$$

Applying this with each character of $\overline{\mathbb{R}}$ in place of f we obtain that $\widehat{\nu}(x) = \lim_{n\to\infty} \widehat{\widehat{\mu_n}}(x) = 2\pi c_{-x}$, if $-x \in A$, and is 0 otherwise, hence $\widehat{\nu}$ has the finite range $2\pi S$. By Theorem 4.1 the set $-A$, and thus A, belongs to the ring of cosets of $\mathbb{R}_d$. At this point we use a theorem of Rosenthal [10] which describes the elements of the ring of cosets of $\mathbb{R}_d$ which are discrete in $\mathbb{R}$. (See also [5].)

Theorem (Rosenthal [10, p. 71]) *The elements of the ring of cosets of* $\mathbb{R}_d$ *which are discrete in the usual topology of* $\mathbb{R}$ *are precisely the sets of the form*

$$F \Delta \bigcup_{j=1}^{k} (\alpha_j \mathbb{Z} + \beta_j) \ , \tag{4.4}$$

where $F \subseteq \mathbb{R}$ *is finite,* $\alpha_j > 0$ *and* $\beta_j \in \mathbb{R}$.

This completes the proof of Theorem 4.2. □

5. Structure Theorem for Tile Sets

We prove a result giving conditions on $f \in L^1(\mathbb{R})$ guaranteeing that all tile sets of bounded density for f are finite unions of complete arithmetic progressions. In §6 we show these conditions include all compactly supported f.

Theorem 5.1. (Structure Theorem) *Let* $f \in L^1(\mathbb{R})$ *have a Fourier transform* $\hat{f}(\xi) \in C^\infty(\mathbb{R})$ *which has a discrete zero set satisfying the bound*

$$\#\{\xi : \ \hat{f}(\xi) = 0 \quad and \ |\xi| \le R\} \le cR \ , \tag{5.1}$$

for some positive constant c. *Suppose that* f *tiles* $\mathbb{R}$ *with the tile set* A *of bounded density. Then the set* A *is a finite union of arithmetic progressions*

$$A = \bigcup_{j=1}^{J} (\alpha_j \mathbb{Z} + \beta_j) \ , \quad \text{all } \alpha_j \neq 0 \ . \tag{5.2}$$

Furthermore every such tiling has a finite partition

$$A = \bigcup_{k=1}^{K} A_k$$

in which each A_k *is a tile set for* f *which is periodic, i.e.* $A_k = \alpha'_k \mathbb{Z} + \{\beta_1^{(k)}, \ldots, \beta_{n_k}^{(k)}\}$.

Proof. We will show that the measure μ_A satisfies the hypotheses of Meyer's Theorem 4.2. By Theorem 3.1 (i) the support of the tempered distribution $\widehat{\mu_A}$ is contained in the set

$$B = \{0\} \cup \{\xi \in \mathbb{R} : \ \hat{f}(\xi) = 0\} \ ,$$

which by hypothesis is a discrete set satisfying (5.1).

Step 1. *The tempered distribution* $\widehat{\mu_A}$ *is a Radon measure.*

Recall that a distribution supported at a single point $b \in \mathbb{R}$ is a finite linear combination of derivatives of δ_b, cf. Rudin [12], Theorem 6.25. Since the support of $\widehat{\mu_A}$ is discrete, we conclude that

$$\widehat{\mu_A} = \sum_{b \in B} \psi_b$$

where

$$\psi_b = \sum_{j=0}^{m_b} c_{b,j} \delta_b^{(j)} .$$

To show that $\widehat{\mu_A}$ is a measure we must show that $m_b = 0$ for all $b \in B$ ($\delta_b^{(j)}$ is the j-th derivative of δ_b, in the sense of distributions; see [12]). Fix $b \in B$, and take a test function $\phi \in C_c^\infty(-1,1)$ such that $\phi^{(j)}(0) = (-1)^j$ for $0 \le j \le m_b$. We consider the distribution $\widehat{\mu_A}$ acting on the scaled test function $g(x) = \phi(\lambda(x-b))$, whose Fourier transform is given by

$$\widehat{g}(\xi) = \frac{1}{\lambda} e^{-i\xi b/\lambda} \widehat{\phi}(\xi/\lambda) ,$$

and let $\lambda \to \infty$. If λ is large enough then the support of $g(x)$ intersects B only at the point b in question. We then have

$$\begin{aligned} \langle g, \widehat{\mu_A} \rangle &= \langle g, \psi_b \rangle \\ &= \sum_{j=0}^{m_b} (-1)^j c_{b,j} g^{(j)}(b) \\ &= \sum_{j=0}^{m_b} (c_{b,j} (-1)^j \phi^{(j)}(0)) \lambda^j \\ &= \sum_{j=0}^{m_b} c_{b,j} \lambda^j . \end{aligned}$$

This is a polynomial in λ of degree m_b. On the other hand

$$\langle g, \widehat{\mu_A} \rangle = \langle \widehat{g}, \widetilde{\mu}_A \rangle = \sum_{a \in A} \widehat{g}(-a) .$$

Now enumerate the points of $A = \{a_n : \ n \ge 1\}$ in increasing order of absolute value.

$$0 \le |a_1| \le |a_2| \le \cdots . \tag{5.3}$$

The bounded density of A implies that $|a_n| \gg n$. Thus, as $n \to \infty$,

$$\begin{aligned} |\widehat{g}(-a_n)| &= \frac{1}{\lambda} \left| \widehat{\phi}(-a_n/\lambda) \right| \ll \frac{1}{\lambda} \left| \frac{-a_n}{\lambda} \right|^{-3/2} \\ &\ll \lambda^{1/2} n^{-3/2} , \end{aligned}$$

where we used $|\widehat{\phi}(\xi)| \ll |\xi|^{-3/2}$. Thus, as $\lambda \to \infty$,

$$\left|\sum_{j=0}^{m_b} c_{b,j}\lambda^j\right| = |\langle g, \widehat{\mu_A}\rangle| \ll \lambda^{1/2} \sum_{n=1}^{\infty} n^{-3/2} \ll \lambda^{1/2} .$$

This cannot happen unless $m_b = 0$. Thus $\widehat{\mu_A}$ is a Radon measure

$$\widehat{\mu_A} = \sum_{b\in B} c_b \delta_b , \tag{5.4}$$

for certain constants $c_b \in \mathbb{C}$.

Step 2. *The coefficients c_b in (5.4) are bounded.*

To prove this, let $\phi \in C_c^\infty(-1,1)$, with $\phi(0) = 1$. Fix $b \in B$, and consider the compactly supported function $h(x) = \phi(\lambda(x-b))$. For large λ the support of $h(x)$ intersects B only at b. We then have

$$c_b = \langle h, \widehat{\mu_A}\rangle = \langle \widehat{h}, \widetilde{\mu}_A\rangle = \sum_{n=1}^{\infty} \widehat{h}(-a_n) ,$$

with $A = \{a_n :\ n \geq 1\}$ as in (5.3).

Since

$$\widehat{h}(\xi) = \frac{1}{\lambda} e^{-i\xi b/\lambda} \widehat{\phi}(\xi/\lambda),$$

we obtain

$$|c_b| \leq \frac{1}{\lambda} \sum_{n=1}^{\infty} \left|\widehat{\phi}(-a_n/\lambda)\right| = S_1 + S_2 ,$$

where S_1 and S_2 represent the sum taken over $0 \leq |a_n| \leq \lambda$ and $|a_n| > \lambda$ respectively. Keeping in mind that the bounded density of A implies $|a_n| \gg n$, we bound S_1 using the estimate $\left|\widehat{\phi}(-a_n/\lambda)\right| \leq \left|\widehat{\phi}\right|_{[-1,1]}$ obtaining

$$S_1 \leq \frac{1}{\lambda} \left|\widehat{\phi}\right|_{[-1,1]} \#\{n :\ |a_n| \leq \lambda\} \ll \frac{1}{\lambda} \cdot \lambda = 1.$$

We estimate S_2 using $\left|\widehat{\phi}(-a_n/\lambda)\right| \ll |-a_n/\lambda|^{-2} \ll \lambda^2/n^2$ to obtain

$$S_2 \leq \frac{1}{\lambda} \sum_{|a_n|>\lambda} \frac{\lambda^2}{n^2} \ll \lambda \sum_{n \gg \lambda} \frac{1}{n^2} \ll \lambda \cdot \frac{1}{\lambda} = 1.$$

Thus $|c_b| \ll 1$.

Step 3. *The set A is a union of doubly infinite arithmetic progressions.*

We are now in a position to apply Meyer's theorem to the measure $\widehat{\mu_A}$. Indeed,

$$|\widehat{\mu_A}|([-R,R]) \le \sup_{b\in B} |c_b| \cdot \#\{b \in B : \ |b| \le R\} \ll R ,$$

by the assumption (5.1) on the zero set of $\widehat{f}(\xi)$. Thus, by Theorem 4.2,

$$A = F \Delta \bigcup_{j=1}^{k} (\alpha_j \mathbb{Z} + \beta_j) ,$$

for some finite set F. If $A' = \bigcup_{j=1}^{k}(\alpha_j\mathbb{Z} + \beta_j)$ then the Poisson summation formula (3.3) implies that $\widehat{\mu}_{A'}$ is a weighted sum of point masses on arithmetic progressions. Then

$$\widehat{\mu_A} = \widehat{\mu_{A'}} + \sum_{n\in F-(A'\cap F)} e^{-in\xi} - \sum_{n\in A'\cap F} e^{-in\xi} .$$

This shows that F must be the empty set, because $\widehat{\mu_A}$ has no continuous part by (5.4). This proves (5.2).

Step 4. *There is a partition $A = \bigcup_{k=1}^{K} A_k$ in which each A_k is a periodic tile set for f.*

Given the decomposition (5.2) of A, call the arithmetic progressions $\alpha_i\mathbb{Z}+\beta_i$ and $\alpha_j\mathbb{Z}+\beta_j$ *equivalent* if α_i/α_j is rational. Group the arithmetic progressions into equivalence classes, and let the sets A_k, $1 \le k \le K$, consist of the union of the arithmetic progressions in each equivalence class. We first show that each A_k is periodic. If $\{\alpha_{j_1}, \ldots, \alpha_{j_m}\}$ denotes the set of periods of the arithmetic progressions in A_k, then each $\alpha_{j_i} = (p_i/q_i)\alpha_{j_1}$. If m_k is the least common multiple of the p_i then A_k is periodic with period $\alpha'_k = m_k\alpha_{j_1}$. (Simply split each arithmetic progression (mod α_{j_i}) into arithmetic progressions (mod α'_k) since $\alpha'_k/\alpha_{j_i} \in \mathbb{Z}$.) Now we may write

$$A_k = \alpha'_k\mathbb{Z} + \{\beta_1^{(k)}, \ldots, \beta_{n_k}^{(k)}\} .$$

It remains to show that each set A_k gives a tiling of $\mathbb{R}$ by f. Define

$$\mu_k = \sum_{a\in A_k} \delta_a , \quad 1 \le k \le K .$$

Using the Poisson summation formula (3.3), the Fourier transform of the measure μ_k is

$$\widehat{\mu}_k = \sum_{n\in\mathbb{Z}} g_k(2\pi n/\alpha'_k)\delta_{2\pi n/\alpha'_k} , \tag{5.5}$$

in which

$$g_k(x) = \frac{2\pi}{|\alpha'_k|} \sum_{l=1}^{n_k} \exp(-i\beta_l^{(k)} x) .$$

In particular each $\widehat{\mu}_k$ is a locally finite measure. Since the ratio of any two distinct α'_k is irrational by construction, the supports of the measures $\widehat{\mu}_k$ are disjoint except at 0.

The Fourier transform $\widehat{\mu_A}$ of μ_A is the sum of the measures $\widehat{\mu}_k$, $1 \leq k \leq K$, hence its support is the union of the supports of the $\widehat{\mu}_k$ except possibly at 0, because none of the weighted point masses in $\widehat{\mu}_k$ can cancel except possibly those centered at 0. Since f tiles $\mathbb{R}$, Theorem 3.1(i) implies that $\widehat{f}(\xi)$ vanishes at all nonzero reals ξ at which $\widehat{\mu_A}$ has mass. These coincide with the nonzero reals ξ at which some $\widehat{\mu}_k$ has mass, whence

$$\operatorname{supp} \widehat{\mu_k} \subseteq \{0\} \ \cup \ \{\xi : \ \widehat{f}(\xi) = 0\} \ ,$$

for $1 \leq k \leq K$. Since each $\widehat{\mu}_k$ is a locally finite measure, Theorem 3.1 (ii) applies to show that each set A_k gives a tiling of $\mathbb{R}$ by f. This completes step 4 and the proof. □

6. Compactly Supported Functions

In this section all functions $f \in L^1(\mathbb{R})$ are assumed to have compact support.

Proof of Theorem 1.1. (i) We show that if $f \in L^1(\mathbb{R})$ has compact support then its Fourier transform $\widehat{f}(\xi)$ satisfies the hypotheses of Theorem 5.1, i.e. $\widehat{f}(\xi) \in C^\infty(\mathbb{R})$ has a discrete zero set satisfying (5.1). If f has support in $[-R, R]$ then the Fourier transform of complex argument

$$\widehat{f}(z) = \int_{-R}^{R} e^{-ixz} f(x) dx \ , \quad z \in \mathbb{C} \ ,$$

is an entire function, which satisfies the growth bound

$$|\widehat{f}(z)| \leq \int_{-R}^{R} e^{xIm(z)} |f(x)| dx \leq ||f||_1 e^{R|z|} \ .$$

Thus $\widehat{f}(z)$ is an entire function of order 1 and type R. If $N(T)$ counts the number of zeros of $\widehat{f}(z)$ in the disk $\{z : \ |z| \leq T\}$, an application of Jensen's formula gives

$$\limsup_{T \to \infty} \frac{N(T)}{T} \leq eR \ ,$$

see Boas [1], Theorem 2.5.13. Thus (5.1) holds for f, so Theorem 5.1 applies to give the decomposition (1.2) of A as a finite union of complete arithmetic progressions.

(ii) To show that A is a finite union of indecomposable tilings, we treat two cases, according to whether $L = \int_{\infty}^{\infty} f(x)dx$ is zero or not. Suppose first that $L \neq 0$. By taking

$-f(x)$ if necessary, we may suppose that $L > 0$. Lemma 2.4 says that the weight of any tiling of bounded density has weight at least $L/(2R)$. We now prove that any tiling of weight w partitions into at most $\lfloor 2Rw/L \rfloor$ indecomposable tilings, by induction on $n \geq 1$, for tilings with weights w satisfying $nL/(2R) \leq w < (n+1)L/(2R)$. For the base case $n = 1$ the tiling A must be indecomposable, else it partitions into tilings $A_1 \cup A_2$, at least one of which has weight less than $L/(2R)$, a contradiction. The induction step is clear, using $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor$.

Suppose next that $L = 0$. Lemma 2.2 says that all tilings A of bounded density have weight 0. Part (i) shows that each such tiling A has an asymptotic density

$$d(A) = \lim_{T \to \infty} \frac{1}{2T} \#\{a \in A : \ |a| \leq T\} \ ,$$

because A is a finite union of arithmetic progressions. We claim that $d(A) \geq 1/(2R)$ for all tilings of bounded density. If not, we can find some $a' \in A$, such that the interval $(a', a' + 2R + \epsilon)$ contains no element of A. There now exists a set U of positive measure in the support of the function $f(x - a')$ which is disjoint from the support of all $f(x - a)$, $a \in A \backslash \{a'\}$. Then

$$\sum_{a \in A} f(x - a) \neq 0 \ \text{ for } x \in U \ ,$$

which contradicts $w = 0$, and this proves the claim. Since $d(A_1 \cup A_2) = d(A_1) + d(A_2)$, we proceed as in the $L > 0$ case to prove any tiling of density $d(A)$ partitions into at most $\lfloor 2R\, d(A) \rfloor$ indecomposable tilings, by induction on n, with the induction step applying to all tilings A having densities $d(A)$ satisfying $n/(2R) \leq d(A) < (n+1)/(2R)$.

Now suppose that A is indecomposable. The last part of Theorem 5.1 says that any bounded density tiling A partitions into a finite union of periodic tilings. Thus A must already be a periodic tiling. □

Proof of Theorem 1.2. Suppose that $f \in L^1(\mathbb{R})$ has compact support and tiles $\mathbb{R}$ with the tile set A of bounded density. Theorem 1.1 (ii) shows that f has a periodic tiling with tile set $A = \alpha\mathbb{Z} + \{\beta_1, \ldots, \beta_J\}$. The Poisson summation formula (3.3) then shows that the measure μ_A has Fourier transform

$$\widehat{\mu_A} = \frac{2\pi}{|\alpha|} \sum_{n \in \mathbb{Z}} \left(\sum_{j=1}^{J} \exp(-2\pi i \beta_j n/\alpha) \right) \delta_{2\pi n/\alpha} \ .$$

Now

$$\operatorname{supp} \widehat{\mu_A} \subseteq \frac{2\pi}{\alpha}\mathbb{Z} ,$$

and supp $\widehat{\mu_A}$ is determined by the set of $n \in \mathbb{Z}$ where the function

$$h(n) = \sum_{j=1}^{J} \theta_j^n, \quad \text{with } \theta_j = e^{-2\pi i \beta_j/\alpha},$$

does not vanish. The Skolem-Mahler-Lech theorem [7, 15] says that the set of integers at which any such exponential polynomial vanishes consists of a finite number of complete arithmetic progressions plus a finite set, which we call the exceptional set. Since $h(0) = J \neq 0$, none of these arithmetic progressions passes through 0. Let ℓ' be the least common multiple of the periods of these complete arithmetic progressions, and let k be the maximal element of the exceptional set if it is nonempty, and let $k = 1$ otherwise. We set $\ell = \ell' k$, and conclude that $h(\ell n) \neq 0$ for all $n \in \mathbb{Z}$. Thus

$$\frac{2\pi\ell}{\alpha}\mathbb{Z} \subseteq \operatorname{supp} \widehat{\mu_A} .$$

Now Theorem 3.1 (i) implies that if $\alpha' = 2\pi\ell/\alpha$ then

$$\alpha'\mathbb{Z}\backslash\{0\} \subseteq \{\xi : \ \widehat{f}(\xi) = 0\} . \tag{6.1}$$

For the reverse direction, suppose that (6.1) holds. Take $A = \alpha\mathbb{Z}$ with $\alpha = 2\pi/\alpha'$. Then

$$\widehat{\mu_A}(\xi) = \frac{2\pi}{|\alpha|} \sum_{n\in\mathbb{Z}} \delta_{n\alpha'} , \tag{6.2}$$

is a locally finite measure, hence Theorem 3.1 (ii) applies to show that f tiles $\mathbb{R}$ using the tile set A.

Proof of Theorem 1.3. It suffices to show that for any set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ there is a function f that tiles $\mathbb{R}$ with each of the sets $A_i = \alpha_i\mathbb{Z}$, for $1 \leq i \leq n$, because any translate of a tile set is itself a tile set. Clearly the function $f_i(x) = \chi_{[0,\alpha_i]}$ tiles $\mathbb{R}$ using A_i, and $\widehat{f_i}$ vanishes on $(2\pi/\alpha_i)\mathbb{Z}\backslash\{0\}$. We choose

$$f = f_1 \star f_2 \star \cdots \star f_n .$$

Then

$$\widehat{f} = \widehat{f_1} \cdot \widehat{f_2} \cdots \widehat{f_n} ,$$

which shows that $\widehat{f}(\xi)$ contains each $(2\pi/\alpha_i)\mathbb{Z}\backslash\{0\}$ in its zero set, for $1 \le i \le n$. However Theorem 1.2 implies that for $1 \le i \le n$, f tiles $\mathbb{R}$ using A_i as a tile set, and the weight is positive since f has positive integral. $\square$

We conclude this section with an application of Theorem 1.1 to the characteristic function χ_S of a set S, to obtain a strengthening of a result of [6].

Theorem 6.1. *Assume that S is a bounded measurable set and that its characteristic function χ_S tiles $\mathbb{R}$ with some weight w. Then w is a positive integer and the tile set is periodic.*

Proof. Since χ_S is nonnegative, all tilings are of bounded density, and since χ_S is a characteristic function, the weight w of any tiling is a positive integer. Theorem 1.1 (ii) says that any such tiling is a finite union of indecomposable tilings, and that all indecomposable tilings are periodic. The period α of any such tile set $A = \alpha\mathbb{Z} + \{\beta_1, \dots, \beta_J\}$ is a rational multiple of L, where $L = \int_{-\infty}^{\infty} \chi_S(t)\, dt = |S|$, since the asymptotic density of A is

$$d(A) = \frac{J}{\alpha} = \frac{w}{L},$$

using Lemma 2.3. Thus each such tiling is periodic with period the least common multiple of the periods of a finite set of indecomposable tilings into which it partitions. $\square$

Theorem 1.1 of [6] proved periodicity of tilings for compact sets S that tile $\mathbb{R}$. Such sets necessarily have a boundary of measure 0. The theorem above includes other measurable sets. For example, let $\mathcal{C} \subseteq [0,1]$ be a Cantor set of positive measure, so that the boundary $\partial\mathcal{C} = \mathcal{C}$, and take $S = \mathcal{C} \cup ([1,2]\backslash(1+\mathcal{C}))$. Then χ_S tiles $\mathbb{R}$ with weight 1 and tile set $\mathbb{Z}$, and S has boundary of positive measure.

7. Examples

We give several examples exhibiting the possible structure of tilings. All examples involve compactly supported functions $f \in L^1(\mathbb{R})$.

Example 7.1. *Functions admitting general tilings of $\mathbb{R}$ that are not of bounded density.*

We give two examples, the first with $\int_{-\infty}^{\infty} f_1(x)dx = 0$, the second with $\int_{-\infty}^{\infty} f_2(x)dx = 1$. The functions $f_j(x)$ are given by

$$f_j(x) = \begin{cases} j & |x| < 1/2, \\ -1/2 & 1/2 \le |x| < 3/2, \\ 0 & |x| > 3/2, \end{cases}$$

for $j = 1, 2$, and are pictured in Figure 7.1.

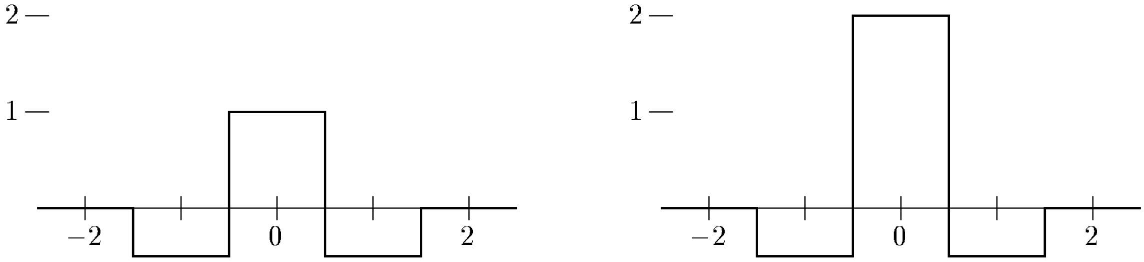

Figure 7.1: f_1 and f_2.

The function $f_1(x)$ tiles $\mathbb{R}$ with weight 0 using the tile set $\mathbb{Z}$ of bounded density. However it also tiles $\mathbb{R}$ with weight -1 using a general tile set A_1 that consists of taking all points $n \in \mathbb{Z}$ with multiplicities $m_1(n) = n^2$. This tile set has polynomial growth

$$T^3 \ll \#\{a \in A : \ |a| \le T\} \ll T^3 \ .$$

hence μ_A is a tempered distribution. The conclusion of Lemma 2.3 (i) fails to hold for the general tile set A_1. In this case the measure $\mu_A = \sum_{n \in \mathbb{Z}} n^2 \delta_n$ has the Fourier transform

$$\widehat{\mu_A} = 2\pi \sum_{n \in \mathbb{Z}} \delta_{2\pi n}^{(2)} \ ,$$

which is a tempered distribution that is not a measure. (Here $\delta_a^{(2)}$ is the second derivative of the point mass δ_a.)

The function $f_2(x)$ tiles $\mathbb{R}$ with weight 1 using the tile set $\mathbb{Z}$ of bounded density. It also tiles $\mathbb{R}$ with weight -1, using a general tile set A_2 that consists of taking all points $n \in \mathbb{Z}$ with multiplicities $m_2(n)$ given by the recursion $m_2(0) = 0$, $m_2(1) = 1$ and

$$m_2(n) = \begin{cases} 4m_2(n-1) - m_2(n-2) + 2, & \text{if } n > 0 \\ m_2(-n), & \text{if } n < 0 \ . \end{cases}$$

The solution to this recursion grows exponentially, with $m_2(n) \approx c(2+\sqrt{3})^n$ as $n \to \infty$. The locally finite measure $\mu_{A_2} = \sum_{n \in \mathbb{Z}} m_2(n) \delta_n$ is a distribution which is not a tempered distribution. The conclusion of Lemma 2.3 (ii) fails to hold for the general tile set A_2.

Example 7.2. *A function f having a weight semigroup $\mathcal{W}(f)$ that is not finitely generated.*

Let χ be the characteristic function of the interval $[-1,1]$, whose Fourier transform is $2\xi^{-1}\sin\xi$. Define a rapidly decreasing sequence $\epsilon_k \to 0$ so that the sequence

$$f_k(x) = \frac{1}{2\epsilon_1}\chi(x/\epsilon_1) \star \cdots \star \frac{1}{2\epsilon_k}\chi(x/\epsilon_k) \tag{7.1}$$

converges uniformly to a function f which is nonnegative and of compact support equal to the interval $[-R,R]$, for $R = \sum_{k=1}^{\infty}\epsilon_k$. (This is certainly possible since, for $k > 1$, the function f_k is uniformly continuous and $f_{k+1}(x)$ is an average of the values of f_k in a small interval around x.) Because of the uniform convergence and the compact support we then have, for each $\xi \in \mathbb{R}$, $\widehat{f}(\xi) = \lim_{k\to\infty}\widehat{f_k}(\xi)$. Since $\widehat{f_k}(\xi)$ is the product of the Fourier transforms of the convolution factors in (7.1) we conclude that $\widehat{f}$ vanishes at the points $\pi n/\epsilon_k$, for all $k = 1, 2, \ldots$, and $n \in \mathbb{Z}\setminus\{0\}$. By Theorem 3.1(ii) f tiles with tile set $A_k = 2\epsilon_k\mathbb{Z}$, for each k. Since $\int_{-\infty}^{\infty} f(t)\,dt = \widehat{f}(0) = 1$ the corresponding weights w_k are equal to $(2\epsilon_k)^{-1}$.

Now choose the numbers ϵ_k so that the set $\{w_k : k = 1, 2, \ldots\}$ is linearly independent over $\mathbb{Q}$. This implies that the semigroup $\mathcal{W}(f)$ requires infinitely many generators.

Example 7.3. *A function having an indecomposable tile set $\alpha\mathbb{Z} + \{\beta_1, \ldots, \beta_J\}$ with integral period α and two cosets β_k and β_l such that $\beta_k - \beta_l$ is irrational.*

Let $f(x) = 1 + \cos 2\pi x$ for $x \in [0,1]$ and let $f(x) = 0$ otherwise. We consider tilings of f with tile sets of the form

$$A = \mathbb{Z} + \{0, \pm t_1, \pm t_2\}, \quad t_1, t_2 \in [0,1). \tag{7.2}$$

For f to tile $\mathbb{R}$ with A it is necessary and sufficient that the function $1 + \cos 2\pi x$ defined on the circle $\mathbb{R}/\mathbb{Z}$ tiles $\mathbb{R}/\mathbb{Z}$ with tile set $\{0, \pm t_1, \pm t_2\}$. Doing harmonic analysis on $\mathbb{R}/\mathbb{Z}$ instead of $\mathbb{R}$, the analogue of Theorem 3.1(i) is that a necessary and sufficient condition for A to tile is that the measure μ on $\mathbb{R}/\mathbb{Z}$ giving unit mass to the five points $0, \pm t_1$, $\pm t_2$, has Fourier transform $\widehat{\mu}$ (defined now on $\mathbb{Z}$, the dual group of $\mathbb{R}/\mathbb{Z}$) whose support is contained in the union of $\{0\}$ and the integer zero set of $\widehat{f}$. Since $\widehat{f}$ vanishes at every integer but at 0 and ± 1 it is enough to ensure that $\widehat{\mu}(\pm 1) = 0$. But

$$\widehat{\mu}(\pm 1) = \int_0^1 e^{-2\pi i x}\,d\mu(x) = 1 + 2\cos 2\pi t_1 + 2\cos 2\pi t_2\ ,$$

and it is clear that, for example, whenever $t_1 \in (1/2, 3/4)$ there is a t_2 such that $\widehat{\mu}(\pm 1) = 0$. Thus there is a large supply of tilings of f of the form (7.2).

Our purpose is to prove the existence of an indecomposable tiling of f with a tile set of the form

$$\alpha\mathbb{Z} + \{\beta_1, \ldots, \beta_J\}\ , \tag{7.3}$$

with the ratio $(\beta_k - \beta_l)/\alpha$ being irrational for some k, l, something which, according to the result of [6], could not happen if f were the characteristic function of a compact set.

Any tiling of f with tile set A as in (7.2) can be written as a finite sum of indecomposable tilings. Each of those is periodic of the form (7.3). Because it is contained in in the tile set (7.2), the period α is necessarily rational, and we can always take α to be an integer by enlarging the number of cosets. The set $\mathcal{B} = \{\beta_1, \ldots, \beta_J\}$ is then constrained to be a subset of $\{0, \ldots, \alpha\} + \{0, \pm t_1, \pm t_2\}$. Consequently there are countably many choices for the set $\mathcal{B}$.

It is clear that f does not tile with any tile set $\subseteq \mathbb{Z}$. Therefore any indecomposable component of the tile set A of (7.3) cannot involve only integers. Thus there is an indecomposable tiling of f with tile set $\mathcal{B}$ which contains an integer m and a number of the form $n + t_j$, where $n \in \mathbb{Z}$ and $j = 1$ or 2, say $j = 1$:

$$\mathcal{B} = \{m, n + t_1, \ldots\}. \tag{7.4}$$

Fix one of the countably many choices for $\mathcal{B}$ and let t_1 vary in the interval $(1/2, 3/4)$ with t_2 defined accordingly from the requirement that $\widehat{\mu}(\pm 1) = 0$. For all but countably many values of t_1 the set $\mathcal{B}$ contains two cosets, namely m and $n + t_1$, whose difference is irrational. Letting $\mathcal{B}$ vary we see that the total exceptional set of t_1's is countable. But t_1 is allowed to vary over a whole interval, thus there is a t_1 for which the tiling (7.2) contains an indecomposable component of integral period and two cosets with irrational difference.

Example 7.4. *A nonnegative function having uncountably many translation-inequivalent indecomposable tilings, all of the same weight.*

The function is the same as in Example 7.3, i.e. $f(x) = 1 + \cos 2\pi x$ for $0 \le x \le 1$ and 0 otherwise. Notice that at least one set $\mathcal{B}$ of the type (7.4) occurs uncountably many times (as t_1 varies) as the set of cosets of an indecomposable tiling of the form (7.3) for a fixed integer α. All these indecomposable tilings have the same weight since that is determined only by α and J of (7.3). There are uncountably many translation-inequivalent tilings among them since the differences of the elements of $\mathcal{B}$ take a finite number of values (mod 1) one of which is t_1.

Example 7.5. *A characteristic function that tiles $\mathbb{R}$ but only with multiple tilings.*

The function $f = \chi_T$ for $T = [0,2] \cup [3,4]$ does not give a tiling of $\mathbb{R}$ of weight 1. However it has a tiling of $\mathbb{R}$ of weight 3 with the tile set $\mathbb{Z}$.

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References

[1] R. P. Boas, *Entire Functions*, Academic Press, New York, 1954.

[2] A.S. Cavaretta, W. Dahmen and C. Micchelli, *Stationary Subdivision*, Memoirs Amer. Math. Soc. **453**, 1991.

[3] P. J. Cohen, On a conjecture of Littlewood and idempotent measures, Amer. J. Math. **82** (1960), 191–212.

[4] A. Córdoba, La formule sommatoire de Poisson, C. R. Acad. Sci. Paris **306**, Série I (1988), 373–376.

[5] A. Córdoba, Dirac Combs, Lett. Math. Phys. **17** (1989), 191–196.

[6] J. C. Lagarias and Y. Wang, Tiling the line with translates of one tile, Inventiones Math., to appear.

[7] C. Lech, A note on recurring series, Ark. Mat. **2** (1953) 417–421.

[8] Y. Meyer, *Nombres de Pisot, nombres de Salem et analyse harmonique*, Lect. Notes Math. **117** (1970), Springer-Verlag.

[9] R. A. Rankin, On the closest packing of spheres in n dimensions, Ann. Math. **48** (1947), 1062-1081.

[10] H. P. Rosenthal, *Projections onto translation-invariant subspaces of $L^P(G)$*, Memoirs Amer. Math. Soc. **63** (1966).

[11] W. Rudin, *Fourier analysis on groups*, Interscience Pubs, New York 1962.

[12] W. Rudin, *Functional Analysis*, McGraw-Hill Inc., New York 1973.

[13] R. S. Strichartz, Construction of Orthonormal Wavelets, in *Wavelets: Mathematics and Applications* (J. J. Benedetto and M. W. Frazier, eds.), CRC Press, Boca Raton 1994, 23–50.

[14] R. Tijdeman, Decomposition of the integers as direct sum of two subsets, in: *Number Theory Seminar, Paris 1992–1993* (ed. S. David), Cambridge U. Press; Cambridge 1995, pp. 261–276.

[15] A. J. van der Poorten, Some facts that should be better known, especially about rational functions, in *Number Theory and Applications*, R. Mollin ed., Kluwer, Dordrecht, 1989, 497-528.

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