THE TURÁN AND DELSARTE PROBLEMS AND THEIR DUALS

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Dedicated to the memory of Jean-Pierre Gabardo

ABSTRACT. We study two optimization problems for positive definite functions on Euclidean space with restrictions on their support and sign: the Turán problem and the Delsarte problem. These problems have been studied also for their connections to geometric problems of tiling and packing. In the finite group setting the weak and strong linear duality for these problems are automatic. We prove these properties in the continuous setting. We also show the existence of extremizers for these problems and their duals, and establish tiling-type relations between the extremal functions for each problem and the extremal measures or distributions for the dual problem. We then apply the results to convex bodies, and prove that the Delsarte packing bound is strictly better than the trivial volume packing bound for *every* convex body that does not tile the space.

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1. Introduction

The Turán extremal problem on positive definite functions with restricted support (see [Ste72] as well as the history and references in [Rev11]) is a very natural question asking how large the integral of a positive definite function f can be if its support is restricted to a domain U and its value at the origin is normalized to be $f(0) = 1$. The problem makes sense in every locally compact abelian (LCA) group and it is interesting in many such groups, including Euclidean spaces (our main focus here) and finite groups. It is easy to see that it suffices to consider continuous positive definite functions, or equivalently, functions f whose Fourier transform $\widehat{f}$ is everywhere nonnegative on the dual group (see [Rud62, Section 1.4]).

If U is a bounded origin-symmetric open set, and A is a measurable set of positive measure such that $A - A \subset U$, then the function $f = m(A)^{-1}\mathbb{1}_A * \mathbb{1}_{-A}$ is supported in U, is positive definite and $f(0) = 1$, hence the supremum in question is at least as large as $\int f = m(A)$. A natural question is therefore to decide if the *Turán constant* of U, defined as

$$T(U) = \sup\left\{ \int f : f(0) = 1, \quad f = 0 \text{ on } U^{\complement}, \quad \widehat{f} \geqslant 0 \right\}, \tag{1.1}$$

is equal to

$$\sup\{m(A) : A - A \subset U\}. \tag{1.2}$$

It is not hard to see that the answer is negative in this generality, i.e. there exist sets U such that (1.1) is strictly greater than (1.2) (examples in finite groups are easy to construct, see e.g. [MR14], and these examples can be carried over to the Euclidean setting in an obvious manner by placing small cubes around points).

However, interestingly, the question remains open if we restrict ourselves to convex domains. If $U \subset \mathbb{R}^d$ is an origin-symmetric convex body, then it is a consequence of the Brunn-Minkowski inequality that the quantity (1.2) is the one that corresponds to the set $A = \frac{1}{2}U$ and hence is equal to $2^{-d}m(U)$. A major open problem is therefore to decide whether $T(U) = 2^{-d}m(U)$ for every origin-symmetric convex body U. We say that U is a *Turán domain* if it satisfies this equality. It is known that any convex body U that tiles the space by translations, as well as the Euclidean ball in $\mathbb{R}^d$, is a Turán domain (and of course their linear images are too) [Gor01, AB02, KR03]. It is also not hard to see that cartesian products of Turán domains are again Turán domains, but we know the quantity $T(U)$ for no other origin-symmetric convex body. Even the simplest cases, such as a regular octagon in the plane, are open.

A related quantity to the Turán constant is the *Delsarte constant* of U, defined as

$$D(U) = \sup\left\{ \int f : f(0) = 1, \;\; f \leqslant 0 \text{ on } U^{\complement}, \;\; \widehat{f} \geqslant 0 \right\}, \quad (1.3)$$

so that the condition $f = 0$ on $U^{\complement}$ in the Turán problem is now replaced with $f \leqslant 0$ on $U^{\complement}$. The Delsarte problem, introduced in [Del72, DGS77], has found many applications in estimating geometric quantities such as sphere packing densities, kissing numbers and more (see [CE03, Via17, CKMRV17, CLS22] and the references in [BR23]).

The Turán and Delsarte extremal problems are optimization problems where we seek the optimum of a linear functional over a function space defined by a set of linear inequalities. In other words, they are *linear programs*, albeit infinite dimensional ones. Their *dual* problems, called the *dual Turán* and *dual Delsarte* problems, are therefore intimately connected to the *primal* Turán and Delsarte problems that we have defined above. These optimization problems are much better understood in the finite group setting, where, importantly, weak and strong duality are always valid. In Section 2 we review the Turán and Delsarte problems along with their duals in finite groups, and establish tiling-type relations among the extremizers of these problems. We also connect the problems to the notions of tiling and spectrality in finite groups.

The extension of many of these results to the continuous setting (of domains in $\mathbb{R}^d$) is not obvious, as in the setting of infinite dimensional linear programs duality may not hold [AN87]. In the literature, results about weak and strong duality and the existence of extremizers have appeared under various conditions [CLS22, BRR24, Gab24]. In this paper we present a unified treatment of the Turán and the Delsarte problems and their duals in the Euclidean setting, and apply our results to obtain interesting consequences concerning the Delsarte packing bound, and properties of Turán domains.

After some preliminaries in Section 3, we define the primal and dual Turán and Delsarte problems in Sections 4 and 5 respectively. We first establish the corresponding weak linear duality inequalities, and then proceed to prove that in fact strong linear duality holds for both problems. We show that extremizers for the Turán and Delsarte problems and their duals exist and satisfy tiling-type convolution equalities.

In Section 6 we connect the Delsarte constant of the difference set $A - A$ to the density of packing by translated copies of a set $A \subset \mathbb{R}^d$, and to tiling and spectrality properties of A. Finally, we apply our results to convex bodies. We prove that the Delsarte packing bound is strictly better than the trivial volume packing bound for every convex body A that does not tile the space. We also give a possible path of attack for proving the existence of a convex body which is not a Turán domain.

2. The Turán and Delsarte problems in finite groups

We start our discussion in the context of finite abelian groups, which motivates the forthcoming results in the Euclidean setting in the later sections. The Turán and Delsarte problems in finite groups were discussed in detail in [MR14]. We recall some relevant results for convenience.

If G is a finite abelian group and $\widehat{G}$ is the dual group, then we define the Fourier transform of a function f on G as

$$\widehat{f}(\gamma) = |G|^{-1/2} \sum_{x \in G} f(x)\gamma(-x), \quad \gamma \in \widehat{G}, \tag{2.1}$$

and the convolution of two functions f and h on G as

$$(f * h)(x) = |G|^{-1/2} \sum_{y \in G} f(y)h(x-y), \quad x \in G. \tag{2.2}$$

The Fourier transforms of $f * h$ and $f \cdot h$ are given by $\widehat{f} \cdot \widehat{h}$ and $\widehat{f} * \widehat{h}$ respectively.

2.1. **The Turán problem.**

Let $U \subset G$ be a set with $0 \in U = -U$, i.e. U is origin-symmetric and contains the origin. A function f on G is called Turán admissible if f is a real-valued function supported on U, $f(0) = 1$ and $\widehat{f}$ is nonnegative. The *Turán constant* $T(U)$ is the supremum of $\widehat{f}(0)$ over all the Turán admissible functions f.

We say that a function h on G is admissible for the dual Turán problem if $h(0) = 1$, h vanishes on $U \setminus \{0\}$, and $\widehat{h}$ is nonnegative. The *dual Turán constant* $T'(U)$ is the supremum of $\widehat{h}(0)$ over all the dual Turán admissible functions h.

It is easy to check using Plancherel's theorem that the inequality $T(U)T'(U) \leqslant 1$ holds, which is referred to as weak linear duality. It was proved in [MR14, Theorem 4.2] that moreover, there is a strong linear duality:

Theorem 2.1. *$T(U)T'(U) = 1$ holds for every $U \subset G$ with $0 \in U = -U$.*

Since the set of Turán admissible functions is compact and the mapping $f \mapsto \widehat{f}(0)$ is continuous, there exists at least one extremal function f for the Turán problem. Similarly, also the dual Turán problem admits an extremal function h.

Proposition 2.2. *If f and h are extremal functions for the Turán problem and its dual respectively, then $f * h = |G|^{-1/2}$ and $\widehat{f} \cdot \widehat{h} = \delta_0$.*

Here and below we use δ_0 to denote the function which takes the value 1 at the origin and vanishes everywhere else. To prove the proposition we observe that

$$1 = \sum_{x \in G} f(x)\overline{h(x)} = \sum_{\gamma \in \widehat{G}} \widehat{f}(\gamma)\widehat{h}(\gamma) \geqslant \widehat{f}(0)\widehat{h}(0) = T(U)T'(U) = 1, \tag{2.3}$$

and as a consequence, the inequality in (2.3) is in fact an equality. Hence $\widehat{f}(\gamma)\widehat{h}(\gamma) = 0$ for all nonzero $\gamma \in \widehat{G}$, which implies the claim.

Remark. We obviously also have $f \cdot h = \delta_0$ and therefore $\widehat{f} * \widehat{h} = |G|^{-1/2}$, but this holds for arbitrary admissible f and h, not only for extremal functions.

2.2. **The Delsarte problem.** We again assume that $U \subset G$ is a set with $0 \in U = -U$. A function f on G is called Delsarte admissible if f is a real-valued function such that $f(0) = 1$, $f(x) \leqslant 0$ for all $x \in G \setminus U$, and $\widehat{f}$ is nonnegative. The *Delsarte constant* $D(U)$ is the supremum of $\widehat{f}(0)$ over all the Delsarte admissible functions f.

We say that a function h on G is admissible for the dual Delsarte problem if $h(0) = 1$, h vanishes on $U \setminus \{0\}$, h is nonnegative on $G \setminus U$, and $\widehat{h}$ is nonnegative everywhere. The *dual Delsarte constant* $D'(U)$ is the supremum of $\widehat{h}(0)$ over all the dual Delsarte admissible functions h.

It is again easy to check that the weak linear duality inequality $D(U)D'(U) \leqslant 1$ holds, while [MR14, Theorem 4.2]) establishes that there is a strong linear duality:

Theorem 2.3. $D(U)D'(U) = 1$ *holds for every* $U \subset G$ *with* $0 \in U = -U$.

The existence of extremal functions for the Delsarte problem and its dual is again obvious.

Proposition 2.4. *If f and h are extremal functions for the Delsarte problem and its dual respectively, then*

(i) $f \cdot h = \delta_0$*, and as a consequence,* $\widehat{f} * \widehat{h} = |G|^{-1/2}$*;*

(ii) $f * h = |G|^{-1/2}$*, and* $\widehat{f} \cdot \widehat{h} = \delta_0$*.*

To prove this we note that

$$1 = f(0)h(0) \geqslant \sum_{x \in G} f(x)h(x) = \sum_{\gamma \in \widehat{G}} \widehat{f}(\gamma)\widehat{h}(\gamma) \qquad (2.4)$$

$$\geqslant \widehat{f}(0)\widehat{h}(0) = D(U)D'(U) = 1, \qquad (2.5)$$

hence both inequalities in (2.4) and (2.5) are equalities. It follows that $f(x)h(x) = 0$ for all nonzero $x \in G$, while $\widehat{f}(\gamma)\widehat{h}(\gamma) = 0$ for all nonzero $\gamma \in \widehat{G}$, which implies both conclusions (i) and (ii) above.

2.3. **Difference sets.** Let $A \subset G$ be an arbitrary nonempty set, and denote

$$m(A) = |G|^{-1/2}|A|. \qquad (2.6)$$

We now connect the Turán and Delsarte constants of the difference set $U = A - A$ to tiling and spectrality properties of the set A. We first observe that the function $f = m(A)^{-1}\mathbb{1}_A * \mathbb{1}_{-A}$ is Turán admissible and $\widehat{f}(0) = m(A)$, hence

$$D(A - A) \geqslant T(A - A) \geqslant m(A). \qquad (2.7)$$

We say that A *tiles by translations* if there is a set $\Lambda \subset G$ such that the translated copies $A + \lambda$, $\lambda \in \Lambda$, form a partition of G. We say that A is a *spectral set* if it admits a system of characters $\Lambda \subset \widehat{G}$ which forms an orthogonal basis for the space $L^2(A)$.

Proposition 2.5. *If A either tiles or is spectral, then*

$$D(A-A) = T(A-A) = m(A). \tag{2.8}$$

Proof. Assume first that A tiles with a translation set $\Lambda \subset G$. Then

$$(A-A) \cap (\Lambda - \Lambda) = \{0\}, \quad |A| \cdot |\Lambda| = |G|. \tag{2.9}$$

Hence the function $h := m(A) \cdot \mathbb{1}_\Lambda * \mathbb{1}_{-\Lambda}$ is dual Delsarte admissible with respect to the set $U = A - A$, and satisfies $\widehat{h}(0) = m(A)^{-1}$. Due to weak linear duality, this implies the inequality $D(A-A) \leqslant m(A)$. Together with (2.7) this implies (2.8).

Next, suppose that A is spectral, i.e. there is a system of characters $\Lambda \subset \widehat{G}$ which forms an orthogonal basis for the space $L^2(A)$. Then

$$\Lambda - \Lambda \subset \{\widehat{\mathbb{1}}_A = 0\} \cup \{0\}, \quad |\Lambda| = \dim L^2(A) = |A|. \tag{2.10}$$

Let h be the function on G defined by

$$h(x) = |A|^{-2} \cdot \Big| \sum_{\gamma \in \Lambda} \gamma(x) \Big|^2, \quad x \in G, \tag{2.11}$$

then both h and $\widehat{h}$ are nonnegative functions,

$$h(0) = 1, \quad \widehat{h}(0) = m(A)^{-1}, \quad \operatorname{supp} \widehat{h} \subset \Lambda - \Lambda. \tag{2.12}$$

This implies that $\widehat{\mathbb{1}}_A \cdot \widehat{h} = \delta_0$, and as a consequence, $\mathbb{1}_A * h = |G|^{-1/2}$. Since $h(0) = 1$ and h is nonnegative, this is possible only if h vanishes on the set $(A-A) \setminus \{0\}$. Hence h is dual Delsarte admissible with respect to $U = A - A$, so again by weak linear duality we obtain that $D(A-A) \leqslant m(A)$. As before, together with (2.7) this yields (2.8). □

Remark. The converse to Proposition 2.5 does not hold. An example constructed in [KLMS24] shows that there is a finite abelian group G and a set $A \subset G$, such that (2.8) holds but A neither tiles nor is spectral.

In the next sections we will see that adapting these results to the Euclidean setting is far from obvious.

3. Euclidean setting preliminaries

In this section we recall some necessary background in the Euclidean setting and fix notation that will be used throughout the paper (see also [Rud91] for more details).

Notation. If $A \subset \mathbb{R}^d$ then $A^{\complement} = \mathbb{R}^d \setminus A$ is the complement of A, ∂A is the boundary of A, and $m(A)$ is the Lebesgue measure of A.

3.1. The *Schwartz space* $S(\mathbb{R}^d)$ consists of all infinitely smooth functions φ on $\mathbb{R}^d$ such that for every n and every multi-index $k = (k_1, \dots, k_d)$, the seminorm

$$\|\varphi\|_{n,k} := \sup_{x \in \mathbb{R}^d} |x|^n |\partial^k \varphi(x)|$$

is finite. A *tempered distribution* is a linear functional on the Schwartz space which is continuous with respect to the topology generated by this family of seminorms. We use $\alpha(\varphi)$ to denote the action of a tempered distribution α on a Schwartz function φ.

We use $S'(\mathbb{R}^d)$ to denote the space of tempered distributions on $\mathbb{R}^d$. A sequence of tempered distributions α_j is said to converge in the space $S'(\mathbb{R}^d)$ if there exists a tempered distribution α such that $\alpha_j(\varphi) \to \alpha(\varphi)$ for every Schwartz function φ.

If φ is a Schwartz function on $\mathbb{R}^d$ then its Fourier transform is defined by

$$\widehat{\varphi}(t) = \int_{\mathbb{R}^d} \varphi(x) e^{-2\pi i \langle t, x \rangle} dx, \quad t \in \mathbb{R}^d.$$

The Fourier transform of a tempered distribution α is defined by $\widehat{\alpha}(\varphi) = \alpha(\widehat{\varphi})$.

If α is a tempered distribution and if φ is a Schwartz function, then the product $\alpha \cdot \varphi$ is a tempered distribution defined by $(\alpha \cdot \varphi)(\psi) = \alpha(\varphi \cdot \psi)$, $\psi \in S(\mathbb{R}^d)$. The convolution $\alpha * \varphi$ of a tempered distribution α and a Schwartz function φ is an infinitely smooth function which is also a tempered distribution, and whose Fourier transform is $\widehat{\alpha} \cdot \widehat{\varphi}$.

A tempered distribution α is called *real* if $\alpha(\varphi)$ is a real scalar for every real-valued $\varphi \in S(\mathbb{R}^d)$. We say that α is *even* if $\alpha(\varphi) = 0$ for every odd $\varphi \in S(\mathbb{R}^d)$.

3.2. If μ is a locally finite (complex) measure on $\mathbb{R}^d$, then we say that μ is a tempered distribution if there exists a tempered distribution α_μ satisfying $\alpha_\mu(\varphi) = \int \varphi d\mu$ for every smooth function φ with compact support. If such α_μ exists, then it is unique.

A measure μ on $\mathbb{R}^d$ is called *translation-bounded* if there exists a constant C such that $|\mu|(B + t) \leqslant C$ for every

$t \in \mathbb{R}^d$, where B is the open unit ball in $\mathbb{R}^d$. If a measure μ is translation-bounded, then it is a tempered distribution.

If μ is a translation-bounded measure on $\mathbb{R}^d$, and if ν is a finite measure on $\mathbb{R}^d$, then the convolution $\mu * \nu$ is a translation-bounded measure.

Lemma 3.1 (see [KL21, Section 2.5]). *Let ν be a finite measure on $\mathbb{R}^d$, and let μ be a translation-bounded measure on $\mathbb{R}^d$ whose Fourier transform $\widehat{\mu}$ is a locally finite measure. Then the Fourier transform of the convolution $\mu * \nu$ is the measure $\widehat{\mu} \cdot \widehat{\nu}$.*

A sequence of measures $\{\mu_j\}$ is said to be *uniformly translation-bounded* if there exists a constant C such that $\sup_t |\mu_j|(B+t) \leqslant C$ for all j, where B is again the open unit ball in $\mathbb{R}^d$. If $\{\mu_j\}$ is a uniformly translation-bounded sequence of measures, then μ_j is said to *converge vaguely* to a measure μ if for every continuous, compactly supported function φ we have $\int \varphi d\mu_j \to \int \varphi d\mu$. In this case, the vague limit μ must also be a translation-bounded measure. From any uniformly translation-bounded sequence of measures $\{\mu_j\}$ one can extract a vaguely convergent subsequence.

Similarly, a sequence of finite measures $\{\mu_j\}$ on $\mathbb{R}^d$ is said to be *bounded* if we have $\sup_j \int |d\mu_j| < +\infty$, and the sequence $\{\mu_j\}$ is said to *converge vaguely* to a finite measure μ if we have $\int \varphi d\mu_j \to \int \varphi d\mu$ for every continuous, compactly supported function φ. Every bounded sequence of measures $\{\mu_j\}$ has a vaguely convergent subsequence.

3.3. We use δ_λ to denote the Dirac measure at the point λ. If $\Lambda \subset \mathbb{R}^d$ is a finite or countable set, then we denote $\delta_\Lambda = \sum_{\lambda \in \Lambda} \delta_\lambda$.

By a *lattice* $L \subset \mathbb{R}^d$ we mean the image of $\mathbb{Z}^d$ under an invertible linear map T. The determinant $\det(L)$ is equal to $|\det(T)|$. The dual lattice L^* is the set of all vectors s such that $\langle l, s \rangle \in \mathbb{Z}$, $l \in L$. The measure δ_L is a tempered distribution, whose Fourier transform is (by Poisson's summation formula) the measure $\widehat{\delta_L} = (\det L)^{-1} \sum_{s \in L^*} \delta_s$.

We say that a set $\Lambda \subset \mathbb{R}^d$ is *locally finite* if the set $\Lambda \cap B$ is finite for every open ball B. We say that Λ is *periodic* if there exists a lattice L such that $\Lambda + L = \Lambda$. If Λ is both locally finite and periodic, then it is a union of finitely many translates of L.

4. The Turán problem and its dual

4.1. Admissible domains for the Turán problem. An open set $U \subset \mathbb{R}^d$ is said to have a *continuous boundary* if for each point $a \in \partial U$ there exist an open ball B centered at a, an orthogonal linear map $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ and a continuous function $\psi : \mathbb{R}^{d-1} \to \mathbb{R}$, such that

$$U \cap B = \{\varphi(x_1, \dots, x_d) : x_d < \psi(x_1, \dots, x_{d-1})\} \cap B. \tag{4.1}$$

This means that locally near each boundary point, the set U consists of those points lying to one side of the graph of some continuous function. This geometric condition is quite general and is satisfied by most domains of practical interest.

We now fix an open set $U \subset \mathbb{R}^d$ with the following properties:

(i) U is an open set of finite measure; (4.2)

(ii) $0 \in U = -U$, that is, U is origin-symmetric and contains the origin; (4.3)

(iii) U has a continuous boundary. (4.4)

Note that the set U may be unbounded, disconnected, or both.

4.2. The Turán constant. A function f on $\mathbb{R}^d$ will be called *Turán admissible* (with respect to the set U) if f is a bounded continuous real-valued function vanishing on $U^\complement$, such that $\widehat{f}$ is nonnegative and $f(0) = 1$.

We first observe that if f is Turán admissible then $f \in L^1(\mathbb{R}^d)$, since f is bounded and vanishes off the set U of finite measure. In turn, since f is continuous and $\widehat{f}$ is nonnegative, it follows that $\widehat{f}$ is a continuous function belonging to $L^1(\mathbb{R}^d)$. Moreover, we have $\|f\|_\infty = \int \widehat{f} = f(0) = 1$. Finally, both f and $\widehat{f}$ are even functions.

Definition 4.1. The *Turán constant* $T(U)$ is the supremum of $\int f$ over all the Turán admissible functions f.

We note that the Turán constant is finite, and in fact, $T(U) \leqslant m(U)$. This is due to the fact that $\|f\|_\infty \leqslant 1$ for any Turán admissible function f.

4.3. Difference sets and convex domains. As an example, suppose that U contains a difference set $A - A$, where $A \subset \mathbb{R}^d$ is a bounded open set. In this case, the function $f = m(A)^{-1}\mathbb{1}_A * \mathbb{1}_{-A}$ is Turán admissible, and $\int f = m(A)$. As a consequence, the lower estimate $T(U) \geqslant m(A)$ holds.

A special case of particular interest is when $U \subset \mathbb{R}^d$ is a *convex* bounded origin-symmetric open set. In this case, U can be realized as the difference set $U = A - A$ where $A = \frac{1}{2}U$. Hence the function $f = m(A)^{-1}\mathbb{1}_A * \mathbb{1}_{-A}$ is Turán admissible, and $T(U) \geqslant m(A) = 2^{-d}m(U)$. We note that any bounded open convex set has a continuous boundary, see [Gri85, Corollary 1.2.2.3], so U satisfies (4.2), (4.3), (4.4).

If a convex bounded origin-symmetric open set $U \subset \mathbb{R}^d$ satisfies $T(U) = 2^{-d}m(U)$, then U is called a *Turán domain*, cf. [KR03]. It is not known whether there exists a convex bounded origin-symmetric open set $U \subset \mathbb{R}^d$ which is not a Turán domain, i.e. such that $T(U) > 2^{-d}m(U)$.

4.4. **The dual Turán constant.** We say that a tempered distribution α on $\mathbb{R}^d$ is admissible for the dual Turán problem, if it is of the form $\alpha = \delta_0 + \beta$, where β is a tempered distribution supported in the closed set $U^\complement$, and moreover α is positive definite, which means that $\widehat{\alpha}$ is a positive measure.

In this case, we may write $\widehat{\alpha} = \widehat{\alpha}(\{0\})\delta_0 + \mu$, where $\widehat{\alpha}(\{0\})$ is the mass of the atom at the origin, and μ is a positive measure on $\mathbb{R}^d$.

Definition 4.2. The *dual Turán constant* $T'(U)$ is the supremum of $\widehat{\alpha}(\{0\})$ over all the tempered distributions α which are admissible for the dual Turán problem.

We first observe that the dual Turán constant $T'(U)$ is a strictly positive number, and in fact, $T'(U) \geqslant m(U)^{-1}$. Indeed, the tempered distribution α given by

$$\alpha = \delta_0 + m(U)^{-1}\mathbb{1}_{U^\complement}, \quad \widehat{\alpha} = m(U)^{-1}\delta_0 + (1 - m(U)^{-1}\widehat{\mathbb{1}}_U), \tag{4.5}$$

is admissible for the dual Turán problem and satisfies $\widehat{\alpha}(\{0\}) = m(U)^{-1}$.

4.5. **Weak linear duality in the Turán problem.** The first result we obtain is the following inequality involving the Turán constant $T(U)$ and its dual $T'(U)$.

Theorem 4.3. *Let $U \subset \mathbb{R}^d$ be an open set satisfying* (4.2), (4.3), (4.4)*. Then*

$$T(U)T'(U) \leqslant 1. \tag{4.6}$$

Note that since $T(U)$ is strictly positive, this shows that $T'(U)$ must be finite.

The inequality (4.6) implies that any tempered distribution α admissible for the dual Turán problem, yields the upper bound $T(U) \leqslant \widehat{\alpha}(\{0\})^{-1}$ for the Turán constant. This principle is usually referred to as *weak linear duality*.

The proof of Theorem 4.3 requires several observations.

4.5.1. First we need the following lemma, based on [MMO14, Lemma 2.4]. Note that the assumption that U has a continuous boundary plays a crucial role in the proof.

Lemma 4.4. *Let f be Turán admissible for an open set U satisfying* (4.2), (4.3), (4.4). *For any $\varepsilon > 0$ there is a smooth real-valued function g with compact support contained in U, $g(0) = 1$, such that $\|\widehat{f} - \widehat{g}\|_1 < \varepsilon$.*

Note that the approximating function g is generally not Turán admissible, since its Fourier transform $\widehat{g}$ need not be a nonnegative function.

Proof of Lemma 4.4. Let ψ be a smooth real-valued function with compact support, $\psi(0) = 1$, such that $\widehat{\psi}$ is nonnegative. Then the function $f_\delta(x) := f(x)\psi(\delta x)$ is Turán admissible, has compact support, and $\|\widehat{f_\delta} - \widehat{f}\|_1 \to 0$ as $\delta \to 0$. Hence, with no loss of generality we may assume that f has compact support.

The closed support of f is thus a compact set K contained in the closure of U. Since U has a continuous boundary, for each point $a \in K \cap \partial U$ there is a small open ball $V(a)$ centered at a, and there is a unit vector $\tau(a)$, such that $K \cap V(a) + \delta\tau(a) \subset U$ for any sufficiently small $\delta > 0$. By compactness we may choose finitely many points $a_1, \dots, a_n \in K \cap \partial U$ such that the open balls $V_j := V(a_j)$, $1 \leqslant j \leqslant n$, cover $K \cap \partial U$.

If we denote $V_0 := U$, then $V_0, V_1, \dots, V_n$ forms an open cover of K. Let $\varphi_0, \dots, \varphi_n$ be a smooth partition of unity subordinate to this open cover, that is, each φ_j is a smooth real-valued function with compact support contained in V_j, and $\sum_{j=0}^{n} \varphi_j(x) = 1$ on K. Hence, if we denote $f_j := f \cdot \varphi_j$

and $\tau_j := \tau(a_j)$, then the function

$$h_\delta(x) = f_0(x) + \sum_{j=1}^{n} f_j(x - \delta\tau_j) \tag{4.7}$$

has compact support contained in U, and $\|\widehat{f} - \widehat{h_\delta}\|_1 \to 0$ as $\delta \to 0$. Moreover, since the origin belongs to the interior of U, we may assume that the open balls $V_1, \dots, V_n$ do not contain the origin, which implies that $h_\delta(0) = f(0) = 1$.

It thus remains to set $g := (h_\delta * \chi)(0)^{-1}(h_\delta * \chi)$, where χ is a smooth nonnegative function supported on a sufficiently small neighborhood of the origin, with $\int \chi = 1$. □

4.5.2. The next observation implies that if a tempered distribution α is admissible for the dual Turán problem, then the measure $\widehat{\alpha}$ is translation-bounded.

Lemma 4.5. *Let α be a tempered distribution on $\mathbb{R}^d$ such that $\alpha = \delta_0$ in some open neighborhood V of the origin, and $\widehat{\alpha}$ is a positive measure. Then the measure $\widehat{\alpha}$ is translation-bounded. Moreover, we have $\sup_t \widehat{\alpha}(B + t) \leqslant C(V)$, where $B \subset \mathbb{R}^d$ is the open unit ball and $C(V)$ is a constant which depends only on V.*

Proof. We choose and fix a smooth function φ with compact support contained in V, and satisfying $\widehat{\varphi}(-t) \geqslant \mathbb{1}_B(t)$ for all $t \in \mathbb{R}^d$. Then

$$\widehat{\alpha}(B + t) = \int \mathbb{1}_B(y - t)d\widehat{\alpha}(y) \leqslant \int \widehat{\varphi}(t - y)d\widehat{\alpha}(y) = (\widehat{\varphi} * \widehat{\alpha})(t), \tag{4.8}$$

due to the positivity of the measure $\widehat{\alpha}$. But note that $\varphi \cdot \alpha = \varphi(0)\delta_0$, which in turn implies that $\widehat{\varphi} * \widehat{\alpha} = \varphi(0)$. Hence the assertion holds with the constant $C(V) = \varphi(0)$. □

4.5.3. Let f be any Turán admissible function, and let α be any tempered distribution admissible for the dual Turán problem. Then $\widehat{\alpha}$ is a positive, translation-bounded measure (due to Lemma 4.5), while $\widehat{f}$ is a nonnegative function in $L^1(\mathbb{R}^d)$. It follows that the convolution $\widehat{f} * \widehat{\alpha}$ is a well-defined, translation-bounded positive measure, which is also a locally integrable function.

Lemma 4.6. *Let f be a Turán admissible function, and let α be an admissible tempered distribution for the dual Turán problem. Then $\widehat{f} * \widehat{\alpha} = 1$ a.e.*

This is obvious if f is a Schwartz function whose closed support is contained in U, since in this case the product $f \cdot \alpha$ is well-defined and is equal to δ_0, and therefore the convolution $\widehat{f} * \widehat{\alpha}$, being the Fourier transform of $f \cdot \alpha$, is the constant function 1.

However, if f is only a continuous function and α is a tempered distribution, then generally the product $f \cdot \alpha$ *does not* make sense. Hence, to prove Lemma 4.6 in the general case, we shall use the approximation result given in Lemma 4.4.

Proof of Lemma 4.6. By Lemma 4.4 there is a sequence of smooth real-valued functions g_j with compact support contained in U, $g_j(0) = 1$, such that $\widehat{g_j} \to \widehat{f}$ in $L^1(\mathbb{R}^d)$. The extra smoothness and support properties of the functions g_j imply that $g_j \cdot \alpha = \delta_0$, and as a consequence, $\widehat{g_j} * \widehat{\alpha}$ is the constant function 1. Now we let $j \to \infty$. Let ψ be a smooth function with compact support. Since $\widehat{\alpha}$ is a translation-bounded measure, the convolution $\widehat{\alpha} * \psi$ is a bounded function. Since $\widehat{g_j} \to \widehat{f}$ in $L^1(\mathbb{R}^d)$, it follows that $\widehat{g_j} * (\widehat{\alpha} * \psi) \to \widehat{f} * (\widehat{\alpha} * \psi)$ pointwise. In turn, this implies that $(\widehat{g_j} * \widehat{\alpha}) * \psi \to (\widehat{f} * \widehat{\alpha}) * \psi$ pointwise, since the convolution is associative (by Fubini's theorem). But $\widehat{g_j} * \widehat{\alpha} = 1$, so we conclude that $(\widehat{f} * \widehat{\alpha}) * \psi = \int \psi$. Since this holds for an arbitrary smooth function ψ with compact support, this shows that $\widehat{f} * \widehat{\alpha} = 1$ a.e. □

Remark. Lemma 4.6 does not hold if we drop the assumption that U has a continuous boundary. An example constructed in [Lev22, Section 3] shows that there is a bounded open set $U \subset \mathbb{R}$ satisfying (4.2) and (4.3), but not (4.4), such that for certain admissible f and α, the function $\widehat{f} * \widehat{\alpha}$ does not coincide a.e. with any constant.

4.5.4. Finally we can establish the weak linear duality inequality (4.6).

Proof of Theorem 4.3. Due to the definitions of the Turán constant $T(U)$ and its dual $T'(U)$, it suffices to verify that if f is a Turán admissible function, and if α is a tempered distribution admissible for the dual Turán problem, then $\widehat{\alpha}(\{0\}) \int f \leqslant 1$.

We may write $\widehat{\alpha} = \widehat{\alpha}(\{0\})\delta_0 + \mu$, where μ is a positive measure. By Lemma 4.6,

$$1 = \widehat{f} * \widehat{\alpha} = \widehat{\alpha}(\{0\})\widehat{f} + \widehat{f} * \mu \geqslant \widehat{\alpha}(\{0\})\widehat{f} \quad \text{a.e.} \tag{4.9}$$

Since $\widehat{f}$ is a continuous function, this implies that the inequality $\widehat{\alpha}(\{0\})\widehat{f}(x) \leqslant 1$ must in fact hold for every $x \in \mathbb{R}^d$. In particular, we have $\widehat{\alpha}(\{0\})\widehat{f}(0) \leqslant 1$, as required. □

4.6. **Strong linear duality in the Turán problem.** Our next goal is to show that the inequality (4.6) is in fact an equality.

Theorem 4.7. *Let $U \subset \mathbb{R}^d$ be an open set satisfying (4.2), (4.3), (4.4). Then*

$$T(U)T'(U) = 1. \tag{4.10}$$

This is usually referred to as *strong linear duality*. This principle also inspires the idea of the proof which will be given next.

4.6.1. Let X be the linear space over $\mathbb{R}$ consisting of all the bounded continuous real-valued and even functions f vanishing on $U^\complement$, and such that $\widehat{f} \in L^1(\mathbb{R}^d)$. We note that if $f \in X$ then also $\widehat{f}$ is a real-valued and even continuous function.

We consider $L^1(\mathbb{R}^d) \times \mathbb{R} \times \mathbb{R}$ as a Banach space over $\mathbb{R}$ (the functions in the first component are taken real-valued) and let K be the subset consisting of all triples

$$\left(\widehat{f} - u, \int f - a, f(0) + b\right) \tag{4.11}$$

where $f \in X$, u is a nonnegative function in $L^1(\mathbb{R}^d)$, and a, b are nonnegative scalars. It is easy to see that the set K is a convex cone.

Lemma 4.8. *The equality (4.10) holds if and only if the triple*

$$(0, T'(U)^{-1}, 1) \tag{4.12}$$

belongs to the closure of K. Moreover, if this is the case then there exists a Turán admissible function f with $\int f = T(U)$.

Proof. In one direction this is obvious: if (4.10) holds then there is a sequence of Turán admissible functions f_j such that $\int f_j \to T'(U)^{-1}$. Then $f_j \in X$, $f_j(0) = 1$, and $\widehat{f_j}$ is a nonnegative function in $L^1(\mathbb{R}^d)$, hence taking $f = f_j$, $u = \widehat{f_j}$, and $a = b = 0$ in (4.11) yields the triple $(0, \int f_j, 1)$ belonging to K whose limit is (4.12).

To prove the converse direction, we suppose that $(\widehat{f_j} - u_j, \int f_j - a_j, f_j(0) + b_j)$ is a sequence in K converging to (4.12), that is,

$$\|\widehat{f_j} - u_j\|_1 \to 0, \quad \int f_j - a_j \to T'(U)^{-1}, \quad f_j(0) + b_j \to 1. \tag{4.13}$$

Since u_j is nonnegative, we have

$$\|u_j\|_1 = \int u_j = f_j(0) - \int (\widehat{f_j} - u_j) \leqslant f_j(0) + b_j + \|\widehat{f_j} - u_j\|_1. \tag{4.14}$$

It follows from (4.13) that the right hand side of (4.14) tends to 1 as $j \to \infty$, hence $\limsup \|u_j\|_1 \leqslant 1$. In particular, $\{u_j\}$ is a bounded sequence in $L^1(\mathbb{R}^d)$, so by passing to a subsequence, we may assume that u_j converges vaguely to some finite measure μ, which ought to be a positive measure since u_j are nonnegative functions.

In turn, we have $\|\widehat{f_j}\|_1 \leqslant \|u_j\|_1 + \|\widehat{f_j} - u_j\|_1$, and so $\limsup \|\widehat{f_j}\|_1 \leqslant 1$. Hence, $\{\widehat{f_j}\}$ is a bounded sequence in $L^1(\mathbb{R}^d)$. Moreover, since we have $\widehat{f_j} - u_j \to 0$ in $L^1(\mathbb{R}^d)$ and therefore also vaguely, the sequence $\{\widehat{f_j}\}$ converges vaguely to the same measure μ.

If we now set $f = \widehat{\mu}$, then f is a bounded continuous function. It follows from the vague convergence that $f_j \to f$ in the sense of tempered distributions. Hence the function f is real-valued and even. Since each f_j vanishes on $U^{\complement}$, then f must be supported in the closure of U, or equivalently, f vanishes in the interior of $U^{\complement}$. Since U has a continuous boundary, the closed set $U^{\complement}$ is equal to the closure of its interior, hence by continuity the function f must in fact vanish in the whole set $U^{\complement}$. Furthermore,

$$\|f\|_\infty = f(0) = \int d\mu \leqslant \limsup \int u_j \leqslant 1, \tag{4.15}$$

and as a consequence, $\|f\|_1 = \int_U |f| \leqslant m(U)$, so $f \in L^1(\mathbb{R}^d)$. It follows that $\mu = \widehat{f}$ is actually a nonnegative continuous function in $L^1(\mathbb{R}^d)$.

We now claim that $\int f_j \to \int f$. Indeed, given $\varepsilon > 0$ we choose a large ball B such that $m(U \setminus B) < \varepsilon$, and let φ be a Schwartz function such that $0 \leqslant \varphi \leqslant 1$, and $\varphi = 1$ on B. Then

$$\int f_j - \int f = \int (f_j - f) \cdot \varphi + \int (f_j - f) \cdot (1 - \varphi). \tag{4.16}$$

Since $\limsup \|f_j\|_\infty \leqslant \limsup \|\widehat{f_j}\|_1 \leqslant 1$, and due to (4.15), for all sufficiently large j the function $(f_j - f) \cdot (1-\varphi)$ is bounded in modulus by an absolute constant C, and it vanishes off the set $U \setminus B$. Hence the second integral on the right hand side of (4.16) is bounded in modulus by $C \cdot m(U \setminus B) < C\varepsilon$. The first integral on the right hand side of (4.16) tends to zero as $j \to \infty$, since $f_j \to f$ in the sense of tempered distributions. This implies that $|\int f_j - \int f| < C\varepsilon$ for all sufficiently large j. As this holds for an arbitrarily small ε, this shows that $\int f_j \to \int f$ and establishes our claim.

Since the scalars a_j are nonnegative, we conclude from (4.13) that

$$\int f = \lim_{j\to\infty} \int f_j \geqslant \lim_{j\to\infty} \left(\int f_j - a_j\right) = T'(U)^{-1}. \tag{4.17}$$

At this point we note that we still do not know that f is Turán admissible, since we have not shown that $f(0) = 1$. However, it follows from (4.15) that $0 \leqslant f(0) \leqslant 1$. Moreover, since $f(0) = \|f\|_\infty$, the value $f(0)$ must be strictly positive, for otherwise this would imply that $f = 0$ which contradicts (4.17).

We have thus shown that $0 < f(0) \leqslant 1$. The function $f(0)^{-1} f$ is therefore Turán admissible, and has integral $f(0)^{-1} \int f \geqslant f(0)^{-1} T'(U)^{-1}$ due to (4.17). On the other hand, we have $f(0)^{-1} \int f \leqslant T(U) \leqslant T'(U)^{-1}$ by the definition of the Turán constant $T(U)$ and the inequality (4.6). This implies that actually $f(0) = 1$ and thus f is a Turán admissible function, and $\int f = T(U) = T'(U)^{-1}$. In particular, the equality (4.10) holds. This completes the proof of Lemma 4.8. ◻

4.6.2. We now continue to the proof of the strong linear duality equality (4.10).

Proof of Theorem 4.7. In view of Lemma 4.8, in order to prove that the equality (4.10) holds, it suffices to show that the triple (4.12) must belong to the closure of K. Suppose to the contrary that this is not the case. Since K is convex, then by the Hahn-Banach separation theorem (see e.g. [Rud91, Theorem 3.4]) there exists a continuous linear functional on the space $L^1(\mathbb{R}^d) \times \mathbb{R} \times \mathbb{R}$ which separates the closure of K from the triple (4.12). This means that there exists an element (g, p, q) of the space

$L^\infty(\mathbb{R}^d)\times\mathbb{R}\times\mathbb{R}$ (again the functions in the first component are taken real-valued), and there is a real scalar c, such that the inequality

$$\int(\widehat{f}-u)g+p\left(\int f-a\right)-q(f(0)+b)\leqslant c \tag{4.18}$$

holds for every $f\in X$, every nonnegative function $u\in L^1(\mathbb{R}^d)$, and every nonnegative scalars a,b, while at the same time we have

$$pT'(U)^{-1}-q>c. \tag{4.19}$$

(The left hand side of (4.19) is the action of (g,p,q) on the triple (4.12).)

We first observe that the function g must be nonnegative a.e. Indeed, if $g(t)<0$ on some set E of positive and finite measure, then taking $f=0$, $a=b=0$ and $u=\lambda\cdot\mathbb{1}_E$ would violate (4.18) for a sufficiently large positive scalar λ. Similarly, p must be a nonnegative scalar, for otherwise taking $f=0$, $u=0$ and $b=0$ would violate (4.18) for a sufficiently large positive scalar a. In the same way also q must be a nonnegative scalar. Thus g is a nonnegative function in $L^\infty(\mathbb{R}^d)$, and p,q are nonnegative scalars.

If we now set $u=0$ and $a=b=0$ in (4.18) then we obtain

$$\int\widehat{f}g+p\int f-qf(0)\leqslant c \tag{4.20}$$

for every $f\in X$. Since X is a linear space, it follows that the left hand side can never be nonzero (for otherwise it could be made positive and arbitrarily large). So we may assume that $c=0$ and that the inequality (4.20) is in fact an equality.

By replacing $g(t)$ with $\frac{1}{2}(g(t)+g(-t))$, we may also assume that g is a nonnegative and even function in $L^\infty(\mathbb{R}^d)$.

If $f\in X$ is a smooth function with compact support contained in U, then the equality in (4.20) may be written in the distributional sense as

$$(\widehat{g}+p-q\delta_0)(f)=c=0. \tag{4.21}$$

Moreover, the tempered distribution $\widehat{g}+p-q\delta_0$ is real and even, therefore the fact that the equality (4.21) holds for all smooth functions $f\in X$ with compact support contained in U, implies that $\widehat{g}+p-q\delta_0$ vanishes in U.

Recall now that p,q are nonnegative scalars. We claim that in fact they are both strictly positive. Indeed, using (4.19) and recalling that $c=0$, it follows that $p>0$. In turn, this implies that $g + p\delta_0$ is a nonzero positive measure, whose Fourier transform satisfies $\widehat{g} + p = q\delta_0$ in U. But since $\widehat{g} + p$ cannot vanish in any neighborhood of the origin, we conclude that also $q > 0$.

Finally, define $\alpha = q^{-1}(\widehat{g} + p)$. Then $\alpha = \delta_0$ in U, and $\widehat{\alpha}$ is a positive measure. Hence α is a tempered distribution admissible for the dual Turán problem. Furthermore, the Fourier transform of α is given by $\widehat{\alpha} = q^{-1}(p\delta_0 + g)$, so that the measure $\widehat{\alpha}$ has an atom at the origin of mass $\widehat{\alpha}(\{0\}) = p/q$. However due to (4.19) and recalling that $c = 0$, we conclude that $\widehat{\alpha}(\{0\}) > T'(U)$, which gives us the desired contradiction. □

4.6.3. *Remark.* We note an interesting consequence of the last proof. Suppose that we consider the following smaller class of admissible tempered distributions α on $\mathbb{R}^d$. We again require that $\alpha = \delta_0 + \beta$, where β is a tempered distribution supported in the closed set $U^\complement$, but in addition we require that $\widehat{\alpha}$ is of the form $\widehat{\alpha} = \widehat{\alpha}(\{0\})\delta_0 + g$, where g is a nonnegative even function in $L^\infty(\mathbb{R}^d)$. Then the supremum of $\widehat{\alpha}(\{0\})$ over this smaller class of admissible α's still gives us the dual Turán constant $T'(U)$.

4.7. **Existence of extremizers for the Turán problem and its dual.** We say that a Turán admissible function f is extremal if it satisfies $\int f = T(U)$. Similarly, a tempered distribution α which is admissible for the dual Turán problem will be called extremal if we have $\widehat{\alpha}(\{0\}) = T'(U)$.

Our next goal is to establish the existence of extremizers for both the Turán problem and its dual. This implies that in the definition of the constants $T(U)$ and $T'(U)$, the supremum is in fact a maximum.

The existence of an extremal function for the Turán problem was proved in [BRR24, Corollary 19] in the more general context of locally compact abelian groups.

Theorem 4.9. *Let $U \subset \mathbb{R}^d$ be an open set satisfying* (4.2), (4.3), (4.4). *Then,*

(i) *The Turán problem admits at least one extremal function f;*

(ii) *The dual Turán problem admits at least one extremal tempered distribution α.*

Proof. Since the equality (4.10) of Theorem 4.7 is now proved, then part (i) follows as a consequence of Lemma 4.8. It thus remains to prove part (ii).

Let $\{\alpha_j\}$ be a sequence of tempered distributions which are admissible for the dual Turán problem, and such that $\widehat{\alpha}_j(\{0\}) \to T'(U)$. It follows from Lemma 4.5 that $\{\widehat{\alpha}_j\}$ is a uniformly translation-bounded sequence of positive measures. As a consequence, by passing to a subsequence we may assume that $\widehat{\alpha}_j$ converges vaguely to some translation-bounded positive measure, which we may denote as $\widehat{\alpha}$ for some tempered distribution α. The uniform translation-boundedness and the vague convergence imply that the sequence $\widehat{\alpha}_j$ converges to $\widehat{\alpha}$ also in the sense of tempered distributions. As a consequence, since $\alpha_j = \delta_0$ in U for all j, then also $\alpha = \delta_0$ in U, hence α is a tempered distribution admissible for the dual Turán problem. Moreover, since the measure $\widehat{\alpha}_j$ is positive and has mass $\widehat{\alpha}_j(\{0\})$ at the origin, then the vague limit $\widehat{\alpha}$ also has an atom at the origin, of mass at least $\lim \widehat{\alpha}_j(\{0\}) = T'(U)$. Hence α is extremal for the dual Turán problem. □

4.8. **Relation between extremizers for the Turán problem and its dual.** Let f and α be extremizers for the Turán problem and its dual, respectively. This means that f is a Turán admissible function, with $\int f = T(U)$, and that α is a tempered distribution admissible for the dual Turán problem, with $\widehat{\alpha}(\{0\}) = T'(U)$. Note that $\widehat{f}$ is a continuous function and $\widehat{\alpha}$ is a measure, so the product $\widehat{f} \cdot \widehat{\alpha}$ is well-defined.

Theorem 4.10. *Let $U \subset \mathbb{R}^d$ be an open set satisfying* (4.2), (4.3), (4.4). *If f and α are any two extremals for the Turán problem and its dual, respectively, then the measure $\widehat{\alpha}$ is supported on the closed set $\{t : \widehat{f}(t) = 0\} \cup \{0\}$, and as a consequence, $\widehat{f} \cdot \widehat{\alpha} = \delta_0$.*

Proof. We can write $\widehat{\alpha} = T'(U)\delta_0 + \mu$, where μ is a positive measure. We claim that μ is supported on the set $\{t : \widehat{f}(t) = 0\}$. Suppose to the contrary that this is not the case, then there is a point a in the closed support of μ such that $\widehat{f}(a) > 0$. Let ψ be a smooth function with compact support, satisfying $0 \leqslant \psi \leqslant \widehat{f}$ and such that $\psi(t) > 0$ in some open neighborhood V of the point a. Since $\widehat{f}$ is an even function, we may assume that ψ is even as well.

Hence $\psi * \mu$ is a nonnegative continuous function with $(\psi * \mu)(0) = \int \psi(t) d\mu(t) > 0$, where the strict inequality holds since the positive measure μ has nonzero mass in V while $\psi(t) > 0$ for $t \in V$. By Lemma 4.6, we have

$$1 = \widehat{f} * \widehat{\alpha} = T'(U)\widehat{f} + \widehat{f} * \mu \quad \text{a.e.} \tag{4.22}$$

The right hand side of (4.22) is therefore a.e. not less than $T'(U)\widehat{f} + \psi * \mu$, which is a nonnegative continuous function whose value at the origin is strictly greater than $T'(U)\widehat{f}(0) = T'(U)T(U) = 1$, which gives us a contradiction. This shows that the measure μ must indeed be supported on the set $\{t : \widehat{f}(t) = 0\}$. In turn, this implies that $\widehat{\alpha}$ is supported on the set $\{t : \widehat{f}(t) = 0\} \cup \{0\}$ and that $\widehat{f} \cdot \widehat{\alpha} = \delta_0$. □

Remark. Note that we *do not* say that $f * \alpha = 1$, since generally if f is a continuous function and α is a tempered distribution, then the convolution $f * \alpha$ *does not* make sense.

5. The Delsarte problem and its dual

5.1. **Admissible domains for the Delsarte problem.**

We now consider a wider class of domains. We fix an open set $U \subset \mathbb{R}^d$ satisfying the following two properties:

(i) U is an open set of finite measure; (5.1)

(ii) $0 \in U = -U$, that is, U is origin-symmetric and contains the origin; (5.2)

Later on, we will also assume that:

(iii) The closed set $U^{\complement}$ is equal to the closure of its interior. (5.3)

The first two conditions (5.1), (5.2) coincide with (4.2), (4.3). The third condition (5.3) will not be assumed from the beginning, since it is not needed in the proof of the weak linear duality. We will impose the condition (5.3) later on, when we establish the strong linear duality and the results which follow it.

The condition (5.3) is a weaker requirement than (4.4). Hence, we consider a more general class of admissible domains for the Delsarte problem. The condition (5.3) is also considered in [BRR24] where it is called “boundary coherence”.

Note again, that the set U may be unbounded, disconnected, or both.

5.2. **The Delsarte constant.** We say that a function f on $\mathbb{R}^d$ is *Delsarte admissible* if f is a continuous real-valued function in $L^1(\mathbb{R}^d)$ satisfying the conditions $f(0) = 1$, $f(t) \leqslant 0$ for $t \in U^\complement$, and $\widehat{f}$ is a nonnegative function.

If f is Delsarte admissible then both f and $\widehat{f}$ are even functions, and $\widehat{f}$ is a continuous function belonging to $L^1(\mathbb{R}^d)$. Moreover, we have $\|f\|_\infty = \int \widehat{f} = f(0) = 1$.

Definition 5.1. The *Delsarte constant* $D(U)$ is the supremum of $\int f$ over all the Delsarte admissible functions f.

The Delsarte constant $D(U)$ is finite, and satisfies $D(U) \leqslant m(U)$.

We observe that $D(U) \geqslant T(U)$, that is, the Delsarte constant is at least as large as the Turán constant, since the supremum is taken over a larger class of admissible functions. Indeed, in the Turán problem we require that f vanishes on $U^\complement$, while in the Delsarte problem f is only required to be nonpositive in $U^\complement$.

In particular (recall Section 4.3) this implies that if U contains a difference set $A - A$, where $A \subset \mathbb{R}^d$ is a bounded open set, then $D(U) \geqslant m(A)$.

5.3. **The dual Delsarte constant.** We say that a tempered distribution α on $\mathbb{R}^d$ is admissible for the dual Delsarte problem, if $\alpha = \delta_0 + \beta$, where β is a positive measure supported in the closed set $U^\complement$, and α is positive definite, which means that $\widehat{\alpha}$ is a positive measure. In this case, we may as before write $\widehat{\alpha} = \widehat{\alpha}(\{0\})\delta_0 + \mu$, where $\widehat{\alpha}(\{0\})$ is the mass of the atom at the origin, and μ is a positive measure.

Definition 5.2. The *dual Delsarte constant* $D'(U)$ is the supremum of $\widehat{\alpha}(\{0\})$ over all the tempered distributions α which are admissible for the dual Delsarte problem.

Similarly, we note that $D'(U) \leqslant T'(U)$, since the supremum is taken over a smaller class of admissible tempered distributions α. Indeed, in the dual Delsarte problem we require β to be a positive measure, while in the dual Turán problem, β is merely a tempered distribution which need not be a measure.

We observe that the tempered distribution α given by (4.5) is admissible for the dual Delsarte problem, hence

$D'(U) \geqslant m(U)^{-1}$. In particular, the dual Delsarte constant is strictly positive.

It follows from Lemma 4.5 that if α is a tempered distribution admissible for the dual Delsarte problem, then $\widehat{\alpha}$ is a translation-bounded measure.

5.4. **Weak linear duality in the Delsarte problem.**

We now turn to prove the inequality that establishes the weak linear duality in the Delsarte problem. Note that this result does not require the condition (5.3).

Theorem 5.3. *Let $U \subset \mathbb{R}^d$ be an open set satisfying* (5.1), (5.2). *Then*

$$D(U)D'(U) \leqslant 1. \tag{5.4}$$

In particular, this shows that the dual Delsarte constant $D'(U)$ is finite.

Moreover, the inequality (5.4) gives us as before, that any tempered distribution α admissible for the dual Delsarte problem yields the upper bound $D(U) \leqslant \widehat{\alpha}(\{0\})^{-1}$.

Proof of Theorem 5.3. Let f be any Delsarte admissible function, and let α be any tempered distribution admissible for the dual Delsarte problem. Note that although f is not necessarily a smooth function, the product $f \cdot \alpha$ is a well-defined signed measure, since f is a continuous function and α is a measure. Moreover, since $\widehat{f}$ is a function in $L^1(\mathbb{R}^d)$ and $\widehat{\alpha}$ is a translation-bounded measure, then by Lemma 3.1 the signed measure $f \cdot \alpha$ is a tempered distribution whose Fourier transform is $\widehat{f} * \widehat{\alpha}$, which is a well-defined, positive translation-bounded measure, and which is also a locally integrable function.

Fix a nonnegative Schwartz function φ with $\int \varphi = 1$, such that $\widehat{\varphi}$ is nonnegative and has compact support. Let $\varphi_\varepsilon(t) = \varepsilon^{-d}\varphi(t/\varepsilon)$, then $\widehat{\varphi}_\varepsilon(x) = \widehat{\varphi}(\varepsilon x)$. The Fourier transform $\widehat{\alpha}$ is of the form $\widehat{\alpha} = \widehat{\alpha}(\{0\})\delta_0 + \mu$, where μ is a positive measure, hence

$$\widehat{f} * \widehat{\alpha} = \widehat{\alpha}(\{0\})\widehat{f} + \widehat{f} * \mu. \tag{5.5}$$

Since φ_ε is nonnegative, and $\widehat{f} * \mu$ is a positive measure, this implies that

$$(\widehat{f} * \widehat{\alpha})(\varphi_\varepsilon) \geqslant \widehat{\alpha}(\{0\}) \int \widehat{f}(t)\varphi_\varepsilon(t)dt \to \widehat{\alpha}(\{0\})\widehat{f}(0) = \widehat{\alpha}(\{0\}) \int f \tag{5.6}$$

as $\varepsilon \to 0$. On the other hand, $\alpha = \delta_0 + \beta$ where β is a positive measure, hence

$$(\widehat{f} * \widehat{\alpha})(\varphi_\varepsilon) = (f \cdot \alpha)(\widehat{\varphi_\varepsilon}) = f(0)\widehat{\varphi_\varepsilon}(0) + \int \widehat{\varphi_\varepsilon}(x) f(x) d\beta(x) \quad (5.7)$$

(the last equality holds and the integral is well-defined since $\widehat{\varphi_\varepsilon}$ has compact support). Since β is a positive measure supported on $U^\complement$, f is nonpositive on $U^\complement$ and $\widehat{\varphi_\varepsilon}$ is nonnegative everywhere, it follows that the integral in (5.7) is a nonpositive scalar. Hence

$$(\widehat{f} * \widehat{\alpha})(\varphi_\varepsilon) \leqslant f(0)\widehat{\varphi_\varepsilon}(0) = 1. \quad (5.8)$$

Combining (5.6), (5.8) yields the inequality $\widehat{\alpha}(\{0\}) \int f \leqslant 1$, which proves (5.4). □

5.5. **Strong linear duality in the Delsarte problem.** Next we show that in fact we have an equality in (5.4), which establishes strong linear duality.

Theorem 5.4. *Let $U \subset \mathbb{R}^d$ be an open set satisfying (5.1), (5.2), (5.3). Then*

$$D(U)D'(U) = 1. \quad (5.9)$$

This result was proved under broader assumptions in [CLS22, Section 3]. Here we give a different presentation following similar lines to our proof of the strong linear duality for the Turán problem (Theorem 4.7).

5.5.1. Let Y be the linear space over $\mathbb{R}$ consisting of all real-valued and even continuous functions f such that both f and $\widehat{f}$ are in $L^1(\mathbb{R}^d)$. Note that if $f \in Y$ then also $\widehat{f}$ is a real-valued and even continuous function.

We now consider $L^1(\mathbb{R}^d) \times L^1(U^\complement) \times \mathbb{R} \times \mathbb{R}$ as a Banach space over $\mathbb{R}$ (the functions in the first and second components are taken real-valued) and let K be the set of all quadruples

$$(\widehat{f} - u, -f|_{U^\complement} - v, \textstyle\int f - a, f(0) + b) \quad (5.10)$$

where $f \in Y$, u is a nonnegative function in $L^1(\mathbb{R}^d)$, v is a nonnegative function in $L^1(U^\complement)$, and a, b are nonnegative scalars. Then the set K is a convex cone.

Lemma 5.5. *The equality (5.9) holds if and only if the quadruple*

$$(0, 0, D'(U)^{-1}, 1) \quad (5.11)$$

belongs to the closure of K. Moreover, if this is the case then there exists a Delsarte admissible function f with $\int f = D(U)$.

Proof. Again one direction is obvious: if (5.9) holds then there is a sequence of Delsarte admissible functions f_j such that $\int f_j \to D'(U)^{-1}$. Then $f = f_j$ is a function in Y, $u = \widehat{f_j}$ is a nonnegative function belonging to $L^1(\mathbb{R}^d)$, $v = -f|_{U^\complement}$ is a nonnegative function in $L^1(U^\complement)$, so together with $a = b = 0$, the quadruple (5.10) becomes $(0, 0, \int f_j, 1)$ which belongs to K and whose limit is (5.11).

We now must prove also the converse direction. Assume that

$$(\widehat{f_j} - u_j, -f_j|_{U^\complement} - v_j, \textstyle\int f_j - a_j, f_j(0) + b_j) \tag{5.12}$$

is a sequence in K converging to (5.11), that is,

$$\begin{aligned} &\int_{\mathbb{R}^d} |\widehat{f_j} - u_j| \to 0, \quad \int_{U^\complement} |f_j + v_j| \to 0, \\ &\int f_j - a_j \to D'(U)^{-1}, \quad f_j(0) + b_j \to 1. \end{aligned} \tag{5.13}$$

In the same way as in the proof Lemma 4.8, we can conclude that $\{u_j\}$ and $\{\widehat{f_j}\}$ are two bounded sequences in $L^1(\mathbb{R}^d)$, with $\limsup \|u_j\|_1$ and $\limsup \|\widehat{f_j}\|_1$ both not exceeding 1, and after passing to a subsequence, these two sequences converge vaguely to a common vague limit which is a finite positive measure μ.

We now show that also $\{f_j\}$ is a bounded sequence in $L^1(\mathbb{R}^d)$. Indeed, we have

$$\|f_j\|_1 = \int_{\mathbb{R}^d} (-f_j) + \int_U (f_j + |f_j|) + \int_{U^\complement} (f_j + |f_j|). \tag{5.14}$$

The first integral $\int(-f_j)$ does not exceed $a_j - \int f_j$ which tends to a limit by (5.13), hence this integral remains bounded from above. The second integral $\int_U (f_j + |f_j|)$ does not exceed $2m(U)\|f_j\|_\infty$ which is bounded due to the inequality $\|f_j\|_\infty \leqslant \|\widehat{f_j}\|_1$ and the fact that $\{\widehat{f_j}\}$ is a bounded sequence in $L^1(\mathbb{R}^d)$. To estimate the third integral $\int_{U^\complement}(f_j + |f_j|)$ we observe that on the set $U^\complement$ we have $f_j \leqslant -v_j + |f_j + v_j|$ as well as the inequality $|f_j| \leqslant v_j + |f_j + v_j|$ (since v_j is a nonnegative function). Hence $f_j + |f_j| \leqslant 2|f_j + v_j|$ and so the integral $\int_{U^\complement}(f_j + |f_j|)$ does not exceed $2\int_{U^\complement} |f_j + v_j|$, which tends to zero by (5.13) and in particular remains bounded.

Thus the right hand side of (5.14) is bounded from above, and $\{f_j\}$ is a bounded sequence in $L^1(\mathbb{R}^d)$.

As a consequence, again by passing to a subsequence we may assume that f_j converges vaguely to a finite signed measure ν. The vague convergence implies that both $f_j \to \nu$ and $\widehat{f_j} \to \mu$ in the sense of tempered distributions. Hence we must have $\widehat{\nu} = \mu$, which implies that in fact both ν and μ are real-valued and even continuous functions belonging to $L^1(\mathbb{R}^d)$. We thus denote $f = \nu$ and note that $\widehat{f} = \mu$ is a nonnegative function.

We now claim that the function f is nonpositive in $U^\complement$. To see this, let ψ be a smooth nonnegative function with compact support contained in the interior of $U^\complement$. Then

$$-\int_{\mathbb{R}^d} f_j\psi = \int_{U^\complement} v_j\psi - \int_{U^\complement} (f_j + v_j)\psi. \tag{5.15}$$

The first integral on the right hand side is nonnegative, while the second integral tends to zero as $j \to \infty$ due to (5.13). Hence using the vague convergence we obtain

$$-\int f\psi = \lim_{j\to\infty}(-\int f_j\psi) \geqslant 0. \tag{5.16}$$

As this holds for an arbitrary smooth nonnegative function ψ with compact support contained in the interior of $U^\complement$, and since f is a continuous function, this implies that f is nonpositive in the interior of $U^\complement$. Since we have assumed that the closed set $U^\complement$ is equal to the closure of its interior, it follows again by the continuity of f that f must be nonpositive in the whole set $U^\complement$.

Next we claim that $\int f \geqslant D'(U)^{-1}$. To prove this, let $\varepsilon > 0$ be given. We choose a large ball B such that $m(U\setminus B) < \varepsilon$, and also $\int_{B^\complement} |f| < \varepsilon$. Let φ be a Schwartz function such that $0 \leqslant \varphi \leqslant 1$, and $\varphi = 1$ on B. We have

$$\int (1-\varphi)\cdot f_j = \int_U (1-\varphi)\cdot f_j + \int_{U^\complement} (1-\varphi)\cdot(f_j+v_j) - \int_{U^\complement} (1-\varphi)\cdot v_j. \tag{5.17}$$

Since $\limsup \|f_j\|_\infty \leqslant \limsup \|\widehat{f_j}\|_1 \leqslant 1$, the function $(1-\varphi)\cdot f_j$ is bounded in modulus by an absolute constant C for all sufficiently large j, and it vanishes on B. Hence the first integral on the right hand side of (5.17) does not exceed $C\cdot m(U\setminus B) < C\varepsilon$. The second integral tends to zero as $j \to \infty$ due to (5.13), while the third integral is nonnegative. This shows that $\limsup \int(1-\varphi)\cdot f_j < C\varepsilon$. Next, we have

$f_j \to f$ in the sense of tempered distributions, hence

$$\int f\cdot\varphi = \lim_{j\to\infty}\int f_j\cdot\varphi = \lim_{j\to\infty}\Big\{\big(\textstyle\int f_j - a_j\big) + a_j - \int (1-\varphi)\cdot f_j\Big\}. \quad (5.18)$$

Since $\int f_j - a_j \to D'(U)^{-1}$, a_j is a nonnegative scalar, and $\limsup \int (1-\varphi)\cdot f_j < C\varepsilon$, this implies that

$$\int f\cdot\varphi > D'(U)^{-1} - C\varepsilon. \quad (5.19)$$

On the other hand, since the function $1-\varphi$ vanishes on B, we have

$$\int f\cdot\varphi = \int f - \int_{B^\complement}(1-\varphi)\cdot f \leqslant \int f + \int_{B^\complement}|f| < \int f + \varepsilon. \quad (5.20)$$

Hence (5.19) and (5.20) yield that $\int f > D'(U)^{-1} - (C+1)\varepsilon$. As this holds for any $\varepsilon > 0$, this shows that indeed $\int f \geqslant D'(U)^{-1}$, and thus our claim is established.

Finally, we show that f is a Delsarte admissible function, that is, we need to establish that $f(0) = 1$. In the same way as in the proof Lemma 4.8, we can show at the first step that we have $0 < f(0) \leqslant 1$ (using the fact that $\limsup \int u_j \leqslant 1$). Hence $f(0)^{-1}f$ is a Delsarte admissible function, whose integral is not less than $f(0)^{-1}D'(U)^{-1}$. But on the other hand, $f(0)^{-1}\int f \leqslant D(U) \leqslant D'(U)^{-1}$ due to the definition of the Delsarte constant $D(U)$ and the inequality (5.4). Hence $f(0) = 1$ and f is Delsarte admissible, and moreover, $\int f = D(U) = D'(U)^{-1}$. In particular, we conclude that the equality (5.9) holds. This completes the proof of Lemma 5.5. □

5.5.2. We continue to the proof of the strong linear duality equality (5.9).

Proof of Theorem 5.4. Due to Lemma 5.5, in order to prove the equality (5.9) it suffices to show that the quadruple (5.11) must belong to the closure of K. Suppose to the contrary that this is not the case. Since K is convex, then the Hahn-Banach separation theorem (see again [Rud91, Theorem 3.4]) yields a continuous linear functional on the space $L^1(\mathbb{R}^d) \times L^1(U^\complement) \times \mathbb{R} \times \mathbb{R}$ which separates the closure of K from the quadruple (5.11), that is, there exists an element (g, h, p, q) of the space $L^\infty(\mathbb{R}^d) \times L^\infty(U^\complement) \times \mathbb{R} \times \mathbb{R}$ (again the functions in the first and second components are taken real-valued), and there is a real scalar c, such that the inequality

$$\int_{\mathbb{R}^d}(\widehat{f}-u)g+\int_{U^\complement}(-f-v)h+p\left(\int f-a\right)-q(f(0)+b)\leqslant c \quad (5.21)$$

holds for every $f \in Y$, every nonnegative function $u \in L^1(\mathbb{R}^d)$, every nonnegative function $v \in L^1(U^\complement)$, and for every nonnegative scalars a, b, while at the same time

$$pD'(U)^{-1}-q>c. \quad (5.22)$$

(The left hand side of (5.22) is the action of (g, h, p, q) on the quadruple (5.11).)

In a similar way as in the proof of Theorem 4.7, we can show that both functions g and h are nonnegative a.e. in their respective domains of definition $\mathbb{R}^d$ and $U^\complement$, and that the scalars p, q are nonnegative. In turn, after setting $u = 0$, $v = 0$ and $a = b = 0$ in the inequality (5.21) it follows (using the fact that Y is a linear space) that

$$\int_{\mathbb{R}^d}\widehat{f}g-\int_{U^\complement}fh+p\int_{\mathbb{R}^d}f-qf(0)=0 \quad (5.23)$$

for every $f \in Y$. Hence we may assume that $c = 0$.

By replacing $g(t)$ with $\frac{1}{2}(g(t)+g(-t))$, and similarly replacing $h(t)$ with $\frac{1}{2}(h(t)+h(-t))$, we may assume that both g and h are nonnegative and even functions on their respective domains of definition $\mathbb{R}^d$ and $U^\complement$ (we note here that $-U^\complement = U^\complement$).

It will be convenient now to extend h to the whole $\mathbb{R}^d$ by setting $h = 0$ on U. In this case, for every *Schwartz* function $f \in Y$ we can write (5.23) as

$$(\widehat{g}-h+p-q\delta_0)(f)=0. \quad (5.24)$$

Using the fact that the tempered distribution $\widehat{g}-h+p-q\delta_0$ is real and even, the equality (5.24) for all Schwartz functions $f \in Y$ implies that $\widehat{g}-h+p-q\delta_0=0$.

Next we show that the scalars p, q are not only nonnegative, but in fact must be strictly positive. Indeed, from (5.22) we obtain that $p > 0$ (since $c = 0$). Hence $g + p\delta_0$ is a nonzero positive measure, whose Fourier transform satisfies $\widehat{g}+p=q\delta_0+h$, and as a consequence, $\widehat{g}+p=q\delta_0$ in U. But since $\widehat{g}+p$ cannot vanish in any neighborhood of the origin, this implies that $q > 0$.

Finally, define $\alpha = q^{-1}(\widehat{g}+p)$. Then $\alpha = \delta_0 + q^{-1}h$, the function $q^{-1}h$ vanishes on U and is nonnegative on $U^\complement$, and $\widehat{\alpha}$ is a positive measure. Hence α is an admissible tempered distribution for the dual Delsarte problem. But $\widehat{\alpha} = q^{-1}(p\delta_0 + g)$, so that $\widehat{\alpha}$ has an atom at the origin of mass $\widehat{\alpha}(\{0\}) = p/q$. However due to (5.22) and recalling that $c = 0$, we have $\widehat{\alpha}(\{0\}) > D'(U)$ which gives us the desired contradiction. □

5.5.3. *Remark.* Again we obtain an interesting fact as a consequence of the proof. We may consider a smaller class of admissible tempered distributions α by requiring that $\alpha = \delta_0 + h$, where $h \in L^\infty(\mathbb{R}^d)$ is a nonnegative even function which vanishes a.e. on U, and that $\widehat{\alpha}$ is of the form $\widehat{\alpha} = \widehat{\alpha}(\{0\})\delta_0 + g$, where g is a nonnegative even function in $L^\infty(\mathbb{R}^d)$. Then the supremum of $\widehat{\alpha}(\{0\})$ over this smaller class of admissible tempered distributions α still gives us the dual Delsarte constant $D'(U)$.

5.6. **Existence of extremizers for the Delsarte problem and its dual.** We say that a Delsarte admissible function f is extremal if it satisfies $\int f = D(U)$. Similarly, a tempered distribution α which is admissible for the dual Delsarte problem is called extremal if $\widehat{\alpha}(\{0\}) = D'(U)$.

The existence of a Delsarte extremizer in the general context of locally compact abelian groups was proved in [Ram25], [BRR24, Corollary 20]. The existence of an extremizer for the dual Delsarte problem is shown in [CLS22, Proposition 3.6].

Theorem 5.6. *Let $U \subset \mathbb{R}^d$ be an open set satisfying* (5.1)*,* (5.2)*,* (5.3)*. Then,*

(i) *The Delsarte problem admits at least one extremal function f;*
(ii) *The dual Delsarte problem admits at least one extremal tempered distribution α.*

Proof. Again part (i) follows from Lemma 5.5 and the equality (5.9) of Theorem 5.4, which is now proved. We therefore turn to prove part (ii).

Let $\{\alpha_j\}$ be a sequence of tempered distributions which are admissible for the dual Delsarte problem, and such that $\widehat{\alpha_j}(\{0\}) \to D'(U)$. Lemma 4.5 implies that $\{\widehat{\alpha_j}\}$ is a uniformly translation-bounded sequence of positive measures. So after passing to a subsequence we may assume that $\widehat{\alpha_j}$ converges vaguely to a certain translation-bounded positive measure, which we denote as $\widehat{\alpha}$ for some tempered distribution α. The uniform translation-boundedness and the vague convergence imply that $\widehat{\alpha}_j \to \widehat{\alpha}$ in the sense of tempered distributions. Hence also $\alpha_j \to \alpha$ in the sense of tempered distributions. Since we have $\alpha_j = \delta_0 + \beta_j$ where β_j is a positive measure supported in the closed set $U^{\complement}$, it follows that the limiting tempered distribution α must also be of the form $\alpha = \delta_0 + \beta$ for some positive measure β supported in the closed set $U^{\complement}$. Hence α is an admissible tempered distribution for the dual Delsarte problem. Moreover, since the measure $\widehat{\alpha}_j$ is positive and has mass $\widehat{\alpha}_j(\{0\})$ at the origin, then the vague limit $\widehat{\alpha}$ also has an atom at the origin, of mass at least $\lim \widehat{\alpha}_j(\{0\}) = D'(U)$. We conclude that α is extremal for the dual Delsarte problem. □

5.7. **Relation between extremizers for the Delsarte problem and its dual.** Our next goal is to establish relations between extremal functions for the Delsarte problem, and extremal tempered distributions for the dual Delsarte problem.

5.7.1. We start with a few general observations. Let f be any function admissible for the Delsarte problem, and let α be any tempered distribution admissible for the dual Delsarte problem (we assume neither f nor α to be extremal here).

First, note that the products $f \cdot \alpha$ and $\widehat{f} \cdot \widehat{\alpha}$ are both well-defined, since f and $\widehat{f}$ are continuous functions, while α and $\widehat{\alpha}$ are positive measures.

Next, we recall that the convolution $\widehat{f} * \widehat{\alpha}$ is well-defined. Indeed, $\widehat{f}$ is a nonnegative continuous function in $L^1(\mathbb{R}^d)$, while $\widehat{\alpha}$ is a positive translation-bounded measure (due to Lemma 4.5), hence $\widehat{f} * \widehat{\alpha}$ is a well-defined, positive translation-bounded measure, which is also a locally integrable function.

We claim that also the convolution $f * \alpha$ is well-defined. Indeed, we have:

Lemma 5.7. *Let α be any tempered distribution admissible for the dual Delsarte problem. Then α is a translation-bounded measure.*

Proof. We choose and fix a Schwartz function φ satisfying $\widehat{\varphi}(t) \geqslant \mathbb{1}_B(t)$ for all $t \in \mathbb{R}^d$, where B is the open unit ball.

Since α is a positive measure, we have

$$\alpha(B+t) \leqslant \int \widehat{\varphi}(s-t)d\alpha(s) = \int e^{2\pi i\langle t,x\rangle}\varphi(x)d\widehat{\alpha}(x), \tag{5.25}$$

and hence

$$\sup_{t\in\mathbb{R}^d} \alpha(B+t) \leqslant \int |\varphi(x)|d\widehat{\alpha}(x) < +\infty, \tag{5.26}$$

where the last integral is finite since φ has fast decay, while $\widehat{\alpha}$ is a translation-bounded measure due to Lemma 4.5. This shows that also α is translation-bounded. ☐

Hence, if f is Delsarte admissible and α is admissible for the dual Delsarte problem, then $f \in L^1(\mathbb{R}^d)$ while α is a positive translation-bounded measure (due to Lemma 5.7), which implies that the convolution $f * \alpha$ is a well-defined translation-bounded signed measure, which is also a locally integrable function.

5.7.2. Now suppose that f and α are extremizers for the Delsarte problem and its dual, respectively. This means that f is a Delsarte admissible function, with $\int f = D(U)$, while α is a tempered distribution which is admissible for the dual Delsarte problem, such that $\widehat{\alpha}(\{0\}) = D'(U)$.

Theorem 5.8. *Let $U \subset \mathbb{R}^d$ be an open set satisfying* (5.1), (5.2), (5.3). *If f and α are any two extremals for the Delsarte problem and its dual, respectively, then*

(i) *The measure α is supported on the closed set $\{x : f(x) = 0\} \cup \{0\}$, and as a consequence, we have $f \cdot \alpha = \delta_0$ and $\widehat{f} * \widehat{\alpha} = 1$ a.e.;*

(ii) *The measure $\widehat{\alpha}$ is supported on the closed set $\{t : \widehat{f}(t) = 0\} \cup \{0\}$, and therefore $\widehat{f} \cdot \widehat{\alpha} = \delta_0$ and $f * \alpha = 1$ a.e.*

Proof. If f and α are extremals then we have $\widehat{\alpha}(\{0\}) \int f = D'(U)D(U) = 1$. Let us recall the proof of Theorem 5.3, and examine the circumstances under which the inequality $\widehat{\alpha}(\{0\}) \int f \leqslant 1$ becomes an equality.

We fix a nonnegative Schwartz function φ with $\int \varphi = 1$, such that $\widehat{\varphi}$ is nonnegative and has compact support. Let $\varphi_\varepsilon(t) = \varepsilon^{-d}\varphi(t/\varepsilon)$, then $\widehat{\varphi}_\varepsilon(x) = \widehat{\varphi}(\varepsilon x)$. We can write $\widehat{\alpha} =$

$D'(U)\delta_0 + \mu$, where μ is a positive measure. Hence

$$(\widehat{f} * \widehat{\alpha})(\varphi_\varepsilon) = D'(U) \int \widehat{f} \cdot \varphi_\varepsilon + \int (\widehat{f} * \mu) \cdot \varphi_\varepsilon. \tag{5.27}$$

On the other hand, $\alpha = \delta_0 + \beta$ where β is a positive measure supported on $U^\complement$. Recalling that the Fourier transform of $f \cdot \alpha$ is $\widehat{f} * \widehat{\alpha}$, due to Lemma 3.1, and since we have $f(0)\widehat{\varphi}_\varepsilon(0) = 1$, this implies that

$$(\widehat{f} * \widehat{\alpha})(\varphi_\varepsilon) = (f \cdot \alpha)(\widehat{\varphi}_\varepsilon) = 1 + \int_{U^\complement} \widehat{\varphi}_\varepsilon(x) f(x) d\beta(x) \tag{5.28}$$

(we also recall that the integral on the right hand side of (5.28) is well-defined, since $\widehat{\varphi}_\varepsilon$ has compact support).

We now claim that the measure β is supported on the closed set $\{x : f(x) = 0\}$, and that the measure μ is supported on the closed set $\{t : \widehat{f}(t) = 0\}$.

To prove these two claims, we first note that β is a positive measure supported on $U^\complement$, while f is a continuous function which is nonpositive on $U^\complement$. Hence $f \cdot \beta$ is a nonpositive measure. Since $\widehat{\varphi}_\varepsilon$ is a nonnegative function, this implies that the right hand side of (5.28) cannot exceed 1. Moreover, since $\widehat{\varphi}_\varepsilon \to 1$ locally uniformly as $\varepsilon \to 0$, it follows that if β is not supported on the closed set $\{x : f(x) = 0\}$, then the $\limsup$ as $\varepsilon \to 0$ of the right hand side of (5.28) must be strictly smaller than 1.

Next, recall that $\widehat{f} * \mu$ is a positive translation-bounded measure, which is also a locally integrable function. Since φ_ε is nonnegative, then $\int (\widehat{f} * \mu) \cdot \varphi_\varepsilon$ is nonnegative; and since $\int \widehat{f} \cdot \varphi_\varepsilon \to \widehat{f}(0) = D(U)$ as $\varepsilon \to 0$, the $\liminf$ as $\varepsilon \to 0$ of the right hand side of (5.27) cannot be less than $D'(U)D(U) = 1$. Moreover, if μ is not supported on the closed set $\{t : \widehat{f}(t) = 0\}$ then, in a similar way to the proof of Theorem 4.10, it can be shown that the function $\widehat{f} * \mu$ is a.e. not less than some nonnegative continuous function whose value at the origin is strictly positive. This implies that the $\liminf$ as $\varepsilon \to 0$ of the right hand side of (5.27) must be strictly larger than 1.

However, since (5.27) and (5.28) are equal, this implies that indeed β is supported on the closed set $\{x : f(x) = 0\}$, and μ is supported on the closed set $\{t : \widehat{f}(t) = 0\}$, for otherwise this would lead to a contradiction.

We conclude that $f \cdot \alpha = \delta_0$ and $\widehat{f} \cdot \widehat{\alpha} = \delta_0$. Finally, since both f and $\widehat{f}$ are in $L^1(\mathbb{R}^d)$, and both α and $\widehat{\alpha}$ are translation-bounded measures, this implies using Lemma 3.1 that $\widehat{f} * \widehat{\alpha} = 1$ a.e., and $f * \alpha = 1$ a.e. Thus both assertions (i) and (ii) are proved. □

6. The Delsarte packing bound, tiling and spectrality

6.1. The essential difference set. Let $A \subset \mathbb{R}^d$ be a bounded measurable set of positive measure. The set

$$\Delta(A) := \{t \in \mathbb{R}^d : m(A \cap (A + t)) > 0\} \tag{6.1}$$

is called the *essential difference set* of A. It is a bounded origin-symmetric open set, that serves as the measure-theoretic analog of the algebraic difference set $A - A$. In particular, if A is an open set, then $\Delta(A) = A - A$.

In this section we connect packing, tiling and spectrality properties of a bounded measurable set $A \subset \mathbb{R}^d$ of positive measure, to the Delsarte problem for the essential difference set $U = \Delta(A)$. We observe that this set U satisfies (5.1) and (5.2), but not necessarily (5.3), as the example $A = (0, 1) \cup (2, 3) \subset \mathbb{R}$ shows.

The Delsarte constant $D(\Delta(A))$ of the set $U = \Delta(A)$ satisfies

$$D(\Delta(A)) \geqslant m(A), \tag{6.2}$$

since the function $f = m(A)^{-1} \mathbb{1}_A * \mathbb{1}_{-A}$ is Delsarte admissible, and $\int f = m(A)$.

6.2. Packing. If $\Lambda \subset \mathbb{R}^d$ is a countable set, then we say that $A + \Lambda$ is a *packing* if the translated copies $A + \lambda$, $\lambda \in \Lambda$, are pairwise disjoint up to measure zero.

Cohn and Elkies [CE03] used the Delsarte problem (not using this name though) as a method for obtaining upper bounds for the density of sphere packings, or more generally, packings by translates of a convex, centrally symmetric body $A \subset \mathbb{R}^d$, see [CE03, Theorem B.1].

The following theorem extends the result to the case where $A \subset \mathbb{R}^d$ is a general bounded measurable set (see also [KR06, Theorem 3] and [BR23, Theorem 1.1]).

Theorem 6.1. *Let $A \subset \mathbb{R}^d$ be a bounded measurable set of positive measure. Then any packing by translates of A has density not exceeding $D(\Delta(A))^{-1}$, that is, the reciprocal of the Delsarte constant of the set $U = \Delta(A)$.*

Hence, any Delsarte admissible (with respect to the set $U = \Delta(A)$) function f, yields an upper bound $(\int f)^{-1}$ for the density of any packing by translates of A. As an example, the function $f = m(A)^{-1}\mathbb{1}_A * \mathbb{1}_{-A}$ yields the trivial volume bound $m(A)^{-1}$.

Note that the proof of Theorem 6.1, as well as Theorems 6.2 and 6.3 below, only relies on Theorem 5.3 which does not require the condition (5.3).

Proof of Theorem 6.1. It is well known and not hard to show that periodic packings come arbitrarily close to the greatest packing density, see [CE03, Appendix A]. Hence it suffices to prove that if Λ is periodic and $A+\Lambda$ is a packing, then the density of Λ cannot exceed $D(\Delta(A))^{-1}$.

Indeed, if Λ is periodic then we may write $\Lambda = L + F$ where L is a lattice and F is a finite set such that $(F - F) \cap L = \{0\}$. Then the measure $\gamma = |F|^{-1}\delta_F * \delta_{-F} * \delta_L$ is translation-bounded, positive and supported on $F-F+L = \Lambda - \Lambda$, which is a subset of $\{0\} \cup \Delta(A)^{\complement}$ by the assumption that $A + \Lambda$ is a packing. Moreover, γ has a unit mass at the origin and $\widehat{\gamma} = |F|^{-1}|\widehat{\delta_F}|^2 \cdot \widehat{\delta_L}$ is a positive measure, so γ is admissible for the dual Delsarte problem. We now observe that $\widehat{\gamma}$ has an atom at the origin of mass $|F| \cdot (\det L)^{-1}$ which is exactly the density of Λ. This shows that the dual Delsarte constant $D'(\Delta(A))$ must be at least as large as the density of Λ. By the weak linear duality inequality (5.4) this implies that the density of Λ cannot exceed $D(\Delta(A))^{-1}$. □

Viazovska [Via17] proved that if A is the open unit ball in $\mathbb{R}^8$, then the Delsarte bound $D(\Delta(A))^{-1}$ coincides with the density of the E_8-lattice packing, showing that this is the densest sphere packing in dimension 8. A similar result was subsequently proved also in dimension 24, see [CKMRV17]. However, note that the Delsarte bound $D(\Delta(A))^{-1}$ is not expected to yield the sharp estimate for sphere packing density in every dimension, and moreover, for certain dimensions the Delsarte bound is known to be not sharp (see [CDV24] and the references therein).

6.3. **Tiling.** We say that a bounded measurable set $A \subset \mathbb{R}^d$ *tiles by translations*, if there is a countable set $\Lambda \subset \mathbb{R}^d$ such that the translated copies $A + \lambda$, $\lambda \in \Lambda$, cover the whole space without overlaps up to measure zero.

The following result is a direct consequence of Theorem 6.1 above (see also [KR06, Section 3.4] and [BR23, Proposition 5.7]).

Theorem 6.2. *If a bounded measurable set $A \subset \mathbb{R}^d$ tiles the space by translations, then we have $D(\Delta(A)) = m(A)$.*

Proof. Indeed, a tiling by translates of A is a packing of density $m(A)^{-1}$. Hence, using Theorem 6.1 we obtain that $m(A)^{-1}$ does not exceed $D(\Delta(A))^{-1}$. But we also have the converse inequality (6.2), so we conclude that the equality $D(\Delta(A)) = m(A)$ holds. ◻

6.4. **Spectrality.** A bounded, measurable set $A \subset \mathbb{R}^d$ is called *spectral* if the space $L^2(A)$ has an orthogonal basis consisting of exponential functions. Fuglede [Fug74] famously conjectured that A is a spectral set if and only if it can tile the space by translations. This conjecture inspired extensive research over the years, see [Kol24] for the history of the problem and an overview of the known related results.

The following result is analogous to Theorem 6.2, but with the tiling assumption being replaced by spectrality (see also [KR06, Theorem 5]).

Theorem 6.3. *If a bounded measurable set $A \subset \mathbb{R}^d$ is spectral, then $D(\Delta(A)) = m(A)$.*

Proof. This is a consequence of a result proved in [LM22, Theorem 3.1]. Stated using the terminology of the present paper, the result asserts that if A is spectral, then there exists a tempered distribution α on $\mathbb{R}^d$, which is admissible for the dual Delsarte problem with respect to the set $U = \Delta(A)$, and such that $\widehat{\alpha}(\{0\}) = m(A)^{-1}$.

This result thus implies that $D'(\Delta(A)) \geqslant m(A)^{-1}$. In turn, as a consequence of the weak linear duality inequality (5.4), it follows that $D(\Delta(A)) \leqslant m(A)$. As before, this suffices to conclude the proof, since the converse inequality (6.2) also holds. ◻

6.5. **Convex bodies.** By a *convex body* $A \subset \mathbb{R}^d$ we mean a compact convex set with nonempty interior. A major recent result proved in [LM22] states that the Fuglede conjecture holds for convex bodies in all dimensions. That is, a convex body $A \subset \mathbb{R}^d$ is a spectral set if and only if it can tile the space by translations.

If $A \subset \mathbb{R}^d$ is a convex body then the set $\Delta(A)$ is convex. Moreover, if the convex body A is origin-symmetric, then $\Delta(A)$ is equal to the interior of the set $2A$.

The next result shows that for a convex body, the converse to Theorems 6.2 and 6.3 is also true: the condition $D(\Delta(A)) = m(A)$ in fact characterizes the convex bodies $A \subset \mathbb{R}^d$ which tile the space by translations (or equivalently, which are spectral).

Theorem 6.4. *Let $A \subset \mathbb{R}^d$ be a convex body. The equality $D(\Delta(A)) = m(A)$ holds if and only if A tiles by translations.*

This result has an interesting consequence for packing density estimates:

Corollary 6.5. *If a convex body $A \subset \mathbb{R}^d$ does not tile the space, then the Delsarte bound $D(\Delta(A))^{-1}$ for the greatest packing density provides a strictly better estimate than the trivial volume packing bound $m(A)^{-1}$.*

It follows that if A does not tile, then there is always a nontrivial packing density estimate (i.e. better than the volume packing bound $m(A)^{-1}$) of the form $(\int f)^{-1}$ for some appropriately chosen Delsarte admissible function f.

The proof of Theorem 6.4 is based on the connection of the problem to the concept of weak tiling, introduced in [LM22] as a relaxation of proper tiling. We say that a bounded, measurable set $A \subset \mathbb{R}^d$ *weakly tiles its complement* if there exists a positive, locally finite measure ν such that $\mathbb{1}_A * \nu = \mathbb{1}_{A^\complement}$ a.e. This notion generalizes proper tilings which correspond to the case where the measure ν is a sum of unit masses.

The notion of weak tiling played a key role in the proof of Fuglede's conjecture for convex domains, due to the fact that every spectral set must weakly tile its complement, see [LM22, Theorem 1.5]. This result can be viewed as a weak form of the "spectral implies tiling" part of Fuglede's conjecture. Note that generally, a set that weakly tiles its complement need not tile properly (as an example, take any spectral set which does not tile).

However, it was proved in [KLM23, Theorem 1.4] that for a convex body, weak tiling implies tiling. More precisely, if a convex body $A \subset \mathbb{R}^d$ weakly tiles its complement, then A must be a convex polytope which can also tile properly by translations. The proof is composed of several results obtained in different papers, see [KLM25, Section 2] for an overview of the ingredients of the proof, the relevant references, and a simplification of part of the proof.

We now turn to show that the condition $D(\Delta(A)) = m(A)$ indeed characterizes the convex bodies $A \subset \mathbb{R}^d$ which tile by translations. Note that if A is a convex body, then the set $U = \Delta(A)$ is a bounded origin-symmetric open convex set, and hence satisfies all the three conditions (5.1), (5.2), (5.3).

Proof of Theorem 6.4. We already know that if A tiles by translations, then the equality $D(\Delta(A)) = m(A)$ holds (Theorem 6.2). We need to prove the converse direction.

Assume that $D(\Delta(A)) = m(A)$. Then the function $f = m(A)^{-1}\mathbb{1}_A * \mathbb{1}_{-A}$ is extremal for the Delsarte problem with $U = \Delta(A)$. By Theorem 5.6(ii), the dual Delsarte problem also admits at least one extremal α. The strong linear duality equality (5.9) gives us that $\widehat{\alpha}(\{0\}) = D'(\Delta(A)) = m(A)^{-1}$. Moreover, by Theorem 5.8(ii), the measure $\widehat{\alpha}$ must be supported on the set $\{t : \widehat{f}(t) = 0\} \cup \{0\}$. However, since $\widehat{f} = m(A)^{-1}|\widehat{\mathbb{1}}_A|^2$, the two functions $\widehat{f}$ and $\widehat{\mathbb{1}}_A$ have the same set of zeros. It thus follows that $\widehat{\mathbb{1}}_A \cdot \widehat{\alpha} = \delta_0$. By Lemma 5.7, the measure α is translation-bounded, so we may use Lemma 3.1 to conclude that $\mathbb{1}_A * \alpha = 1$ a.e. In turn, since we have $\alpha = \delta_0 + \beta$ for some positive measure β, this implies that $\mathbb{1}_A * \beta = \mathbb{1}_{A^\complement}$ a.e., that is, A weakly tiles its complement. Finally, [KLM23, Theorem 1.4] yields that A can also tile properly by translations. □

6.6. **Turán domains.** Lastly, we mention another interesting consequence of Theorem 6.4 in relation to the possible existence of non-Turán domains.

First we recall that for a convex bounded origin-symmetric open set $U \subset \mathbb{R}^d$ we have

$$D(U) \geqslant T(U) \geqslant 2^{-d}m(U). \tag{6.3}$$

The last inequality is true since the closure of the set $\frac{1}{2}U$ is a convex body A satisfying $\Delta(A) = U$ and $m(A) = 2^{-d}m(U)$, and so the function $f = m(A)^{-1}\mathbb{1}_A * \mathbb{1}_{-A}$ is Turán admissible and satisfies $\int f = 2^{-d}m(U)$.

As an example, let $U \subset \mathbb{R}^d$ be an open ball centered at the origin. In this case, it is known that $T(U) = 2^{-d}m(U)$, see [Gor01], [KR03], [Gab24]. On the other hand, in dimensions greater than one, a ball cannot tile by translations. Hence Theorem 6.4 yields that $D(U) > T(U) = 2^{-d}m(U)$, that is, the Delsarte constant of a ball is strictly greater than the corresponding Turán constant.

Recall that it is not known whether there exists a convex bounded origin-symmetric open set $U \subset \mathbb{R}^d$ which is not a Turán domain, i.e. such that $T(U) > 2^{-d}m(U)$. The following result gives a possible line of attack for constructing a non-Turán domain.

Corollary 6.6. *Assume that there exists a convex bounded origin-symmetric open set $U \subset \mathbb{R}^d$, which does not tile, but $T(U) = D(U)$. Then U is a non-Turán domain.*

Proof. Indeed, if U does not tile then by Theorem 6.4 we have $D(U) > 2^{-d}m(U)$, so using the assumption that $T(U) = D(U)$ this implies that U is a non-Turán domain. ☐

In conclusion, for a convex body $A \subset \mathbb{R}^d$, it is instructive to inspect the chain of inequalities

$$m(A) \leqslant 2^{-d}m(\Delta(A)) \leqslant T(\Delta(A)) \leqslant D(\Delta(A)). \qquad (6.4)$$

If A is non-symmetric then the first inequality is strict by the Brunn-Minkowski inequality. If A is symmetric but does not tile the space, then the first inequality becomes an equality, but at least one of the other two inequlities must be strict by Theorem 6.4. Finally, if A tiles the space, then all inequalities become equalities.

7. Remarks

7.1. Some authors define the Turán constant alternatively as the supremum of $\int f$ over all the continuous real-valued functions f, *with compact support contained in the open set U*, such that $f(0) = 1$ and $\widehat{f}$ is nonnegative. We denote this supremum by $T_0(U)$.

It is obvious that $T_0(U) \leqslant T(U)$. The question whether the equality $T_0(U) = T(U)$ holds or not, seems to be largely open. It is easy to show that the equality holds if $U \subset \mathbb{R}^d$ is a convex bounded origin-symmetric open set, or more generally, if $U \subset \mathbb{R}^d$ is a bounded origin-symmetric open set assumed to be *strictly star-shaped* (with respect to the origin), which by definition means that the closure of λU is contained in U for every $0 \leqslant \lambda < 1$, see e.g. [Mav13, Theorem 1].

Another case where the equality is known to hold is when $U \subset \mathbb{R}$ is a bounded open set, $0 \in U = -U$, composed of finitely many intervals, see [Mav13, Theorem 2].

No example seems to be known of an open set $U \subset \mathbb{R}^d$ of finite measure, $0 \in U = -U$, such that $T_0(U) < T(U)$.

See also [BRR24, Section 5] where the question is discussed in the general context of locally compact abelian groups.

7.2. The Turán problem and its dual were also considered by Gabardo in [Gab24]. The main result [Gab24, Theorem 4] concerns the case where U is an open ball (centered at the origin) in $\mathbb{R}^d$. In this case, Gabardo constructed a tempered distribution α which is admissible for the dual Turán problem, such that $\widehat{\alpha}(\{0\}) = 2^d m(U)^{-1}$. This allowed him to conclude that α is extremal for the dual Turán problem, and to obtain a new proof of the fact that if U is an open ball in $\mathbb{R}^d$ then $T(U) = 2^{-d} m(U)$. It is also noted [Gab24, Proposition 18] that this extremal tempered distribution α is *not a measure*.

In [Gab19], [Gab20], Gabardo announced a result concerning the factorization of positive definite functions through convolutions in locally compact abelian groups, which implies that the strong linear duality equality $T(U)T'(U) = 1$ holds for any open set $U \subset \mathbb{R}^d$ of finite measure, $0 \in U = -U$. To our knowledge, the proof remains unpublished.

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