

STRUCTURE RESULTS FOR MULTIPLE TILINGS IN 3D

NICK GRAVIN, MIHAIL N. KOLOUNTZAKIS, SINAI ROBINS, AND DMITRY SHIRYAEV

1. INTRODUCTION

The study of multiple tilings of Euclidean space began in 1936, when the famous Minkowski facet-to-facet conjecture [18] for classical tilings was extended to the setting of k -tilings with the unit cube, by Furtwängler [4]. Minkowski's facet-to-facet conjecture states that for any *lattice* tiling of \mathbb{R}^d by translations of the unit cube, there exist at least two translated cubes that share a facet (face of co-dimension 1). This conjecture was strengthened by Keller [10] who conjectured the same conclusion for *any* cube tiling, not just lattice tilings. It was also strengthened in a different direction by Furtwängler [4] who, again, conjectured the same conclusion for any *multiple* lattice tiling.

To define a multiple tiling, suppose we translate a convex body P with a discrete multiset Λ , in such a way that each point of \mathbb{R}^d gets covered exactly k times, except perhaps the translated copies of the boundary of P . We then call such a body a k -tiler, and such an action has been given the following names in the literature: a **k -tiling**, a **tiling at level k** , a **tiling with multiplicity k** , and sometimes simply a **multiple tiling**. We may use any of these synonyms here, and we immediately point out, for polytopes P , a trivial but useful algebraic equivalence for a tiling at level k :

$$(1) \quad \sum_{\lambda \in \Lambda} \mathbf{1}_{P+\lambda}(x) = k,$$

for almost all $x \in \mathbb{R}^d$, where $\mathbf{1}_P$ is the indicator function of the polytope P .

Furtwängler's conjecture was disproved by Hajós [8] for dimension larger than 3 and for $k \geq 9$ while Furtwängler himself [4] proved it for dimension at most 3. Hajós [9] also proved Minkowski's conjecture in all dimensions. The ideas of Furtwängler were subsequently extended (but still restricted to cubes) by the important work of Perron [19], Robinson [20], Szabó [24], Gordon [5] and Lagarias and Shor [14]. These authors showed that for some levels k and dimensions d and under the lattice assumption as well as not, a facet-to-facet conclusion for k -tilings is true in \mathbb{R}^d , while for most values of k and d it is false.

There is a vast literature on the study of coverings of Euclidean space by a convex body, and an equally vast body of work on classical tilings by translations of one convex body, which must necessarily be a polytope (see for example [3, 7]). On the one hand, when we consider a k -tiling polytope P , we obtain an exact covering of \mathbb{R}^d , in the sense that almost every point of \mathbb{R}^d gets covered exactly k times. On the other hand, the family of k -tilers is much larger than the family of 1-tilers. Hence the study of k -tilings lies somewhere between coverings and 1-tilings.

It was known to Bolle [2] that in \mathbb{R}^2 , every k -tiling convex polytope has to be a centrally symmetric polygon, and using combinatorial methods Bolle [2] gave a characterization for all polygons in \mathbb{R}^2 that admit a k -tiling with a *lattice* Λ of translation vectors. Kolountzakis [13] proved that if a convex polygon P tiles \mathbb{R}^2 multiply with *any* discrete multiset Λ , then Λ must be a *finite union of two-dimensional lattices*. The ingredients of Kolountzakis' proof include the idempotent theorem for the Fourier transform of a measure. Roughly speaking, the idempotent

Date: August 6, 2012.

M. K.: Supported by research grant No 3223 from the Univ. of Crete and in part by the Singapore MOE Tier 2 research grant MOE2011-T2-1-090.

N. G., S. R., D. S.: Supported in part by the Singapore MOE Tier 2 research grant MOE2011-T2-1-090.

Keywords: Multiple tilings; Tilings; Lattices; Fourier Transform; Zonotope; Quasi-periodicity.

theorem of Meyer [17] tells us that if the square of the Fourier transform of a measure is itself, then the support of the measure is contained in a finite union of lattices. To put our main result into its proper context, we record here the precise result of Kolountzakis. A multiple tiling is called **quasi-periodic** if its multiset of discrete translation vectors Λ is a finite union of translated lattices, not necessarily all of the same dimension.

Theorem (Kolountzakis, 2002 [11]). *Suppose that K is a symmetric convex polygon which is not a parallelogram. Then K admits only quasi-periodic multiple tilings if any.*

Here we extend this result to \mathbb{R}^3 , and we also find a fascinating class of polytopes analogous to the parallelogram of the theorem above. To describe this class, we first recall the definition of a **zonotope**, which is the Minkowski sum of a finite number of line segments. In other words, a zonotope equals a translate of $[-\mathbf{v}_1, \mathbf{v}_1] + \cdots + [-\mathbf{v}_N, \mathbf{v}_N]$, for some positive integer N and vectors $\mathbf{v}_1, \dots, \mathbf{v}_N \in \mathbb{R}^d$. A zonotope may equivalently be defined as the projection of some l -dimensional cube. A third equivalent condition is that for a d -dimensional zonotope, all of its k -dimensional faces are centrally symmetric, for $1 \leq k \leq d$. For example, the zonotopes in \mathbb{R}^2 are the centrally symmetric polygons.

We shall say that a polytope $P \subseteq \mathbb{R}^3$ is a **two-flat zonotope** if P is the Minkowski sum of $n + m$ line segments which lie in the union of two different two-dimensional subspaces H_1 and H_2 . In other words, H_1 contains n of the segments and H_2 contains m of the segments (if one of the segments belongs to both H_1 and H_2 we list it twice, once for each plane). Equivalently, P may be thought of as the Minkowski sum of two 2-dimensional symmetric polygons one of which may degenerate into a single line segment.

Recently, a structure theorem for convex k -tilers in \mathbb{R}^d was found, and is as follows.

Theorem (Gravin, Robins, Shiryaev 2011 [6]). *If a convex polytope k -tiles \mathbb{R}^d by translations, then it is centrally symmetric and its facets are centrally symmetric.*

In the present context of \mathbb{R}^3 , it follows immediately from the latter theorem that a k -tiler $P \subset \mathbb{R}^3$ is necessarily a zonotope. In this paper we extend the result of Kolountzakis [11] from \mathbb{R}^2 to \mathbb{R}^3 , providing a structure theorem for multiple tilings by polytopes in three dimensions.

Main Theorem. *Suppose a polytope P k -tiles \mathbb{R}^3 with a discrete multiset Λ , and suppose that P is not a two-flat zonotope. Then Λ is a finite union of translated lattices.*

It turns out that if P is a rational two-flat zonotope, then P admits a k -tiling with a non-quasi-periodic set of translation vectors Λ , as we show in our Corollary 7.1. For some of the classical study of 1-tilings, and their interesting connections to zonotopes, the reader may refer to the work of [15], [16], [22], [25], and [1]. Here we find it very useful to use the intuitive language of distributions [21, 23] in order to think – and indeed discover – facts about k -tilings. To that end we introduce the distribution (which is locally a measure)

$$(2) \quad \delta_\Lambda := \sum_{\lambda \in \Lambda} \delta_\lambda,$$

where δ_λ is the Dirac delta function at $\lambda \in \mathbb{R}^d$. To develop some intuition, we may check formally that

$$\delta_\Lambda * \mathbf{1}_P = \sum_{\lambda \in \Lambda} \delta_\lambda * \mathbf{1}_P = \sum_{\lambda \in \Lambda} \mathbf{1}_{P+\lambda},$$

so that from the first definition (1) of k -tiling, we see that a polytope P is a k -tiler if and only if

$$(3) \quad \delta_\Lambda * \mathbf{1}_P = k.$$

The paper is modularized into short sections that highlight each step separately, and the organization runs as follows. In §3 we compute the Fourier transform of any 4-legged frame of a polytope, and show that its zeros form a certain countable union of hyperplanes. In §4 we find a sufficient condition, which we call the *intersection property*, for the Fourier transform of δ_Λ to have a discrete support. Then we show that if P is a k -tiler, and if the intersection property

holds for all 4-legged frames of P , then $\text{supp } \widehat{\delta}_\Lambda$ (the support of the distribution $\widehat{\delta}_\Lambda$, the Fourier Transform of the distribution δ_Λ) is a discrete set in \mathbb{R}^3 , of bounded density.

In §5, we prove that the intersection property implies the quasi-periodicity of Λ .

In §6 we discover a fascinating family of k -tilers, associated to a non-discrete $\text{supp } \widehat{\delta}_\Lambda$. We prove that if P tiles \mathbb{R}^3 with multiplicity, by translations with a discrete multiset Λ , and the intersection property fails to hold, then P must be a two-flat zonotope.

The proof of the Main Theorem is also given in §6; this proof is quite short since it just strings together all of the results of the previous sections. In the final section, we show that each rational two-flat zonotope admits a very peculiar non-quasi-periodic k -tiling. We note that it may be quite difficult to offer a visualization of these 3-dimensional non-quasi-periodic tilings, and that we discovered them by using Fourier methods.

2. PRELIMINARIES

Suppose the polytope P tiles multiply with the translates $\Lambda \subseteq \mathbb{R}^d$. We will need to understand some basic facts about how the Λ points are distributed, for example in Theorem 5.1 below.

Definition 2.1. (*Uniform density*)

A multiset $\Lambda \subseteq \mathbb{R}^d$ has asymptotic density ρ if

$$\lim_{R \rightarrow \infty} \frac{\#(\Lambda \cap B_R(x))}{|B_R(x)|} \rightarrow \rho$$

uniformly in $x \in \mathbb{R}^d$. In this case we write $\rho = \text{dens } \Lambda$. Another (weaker) notion is that of bounded density - we say that Λ has (uniformly) bounded density if $\frac{\#(\Lambda \cap B_R(x))}{|B_R(x)|} \leq M$ for $x \in \mathbb{R}^d$, and $R > 1$. We say then that Λ has density (uniformly) bounded by M .

Since we intend to speak of the Fourier Transform of δ_Λ it is important to us that if Λ has bounded density then δ_Λ is a tempered distribution [21, 23], and, therefore, its Fourier Transform $\widehat{\delta}_\Lambda$ is well defined. And, it is almost obvious by comparing volumes that, if a polytope P k -tiles \mathbb{R}^d with translates Λ then Λ has density $k/|P|$.

For any symmetric polytope P , and any face $F \subset P$, we define F^- to be the face of P symmetric to F with respect to P 's center of symmetry. We call F^- the **opposite face** of F .

Throughout the paper, we use the notation \mathbf{x}^\perp to denote the perpendicular subspace, of codimension 1, to the vector \mathbf{x} . We also use the standard convention of boldfacing all vectors, to differentiate between \mathbf{v} and v , for example. We furthermore use the convention that $[\mathbf{e}]$ denotes the 1-dimensional line segment from 0 to the endpoint of the vector \mathbf{e} . Whenever it is clear from context, we will also write $[e]$ to denote the same line segment - for example, in the case that e denotes an edge of a polytope.

We let $Z(f)$ be the zero set of the function f .

Definition 2.2 (4-legged-frame of a polytope).

(a) Suppose $P \in \mathbb{R}^3$ is a zonotope (symmetric polytope with symmetric facets). A collection of four (one-dimensional) edges of P is called a **4-legged-frame** if whenever e is one of the edges then there exist two vectors τ_1 and τ_2 such that the four edges are

$$[e], [e] + \tau_1, [e] + \tau_2 \text{ and } [e] + \tau_1 + \tau_2,$$

and such that the edges $[e]$ and $[e] + \tau_1$ belong to the same face of P and the edges $[e] + \tau_2$ and $[e] + \tau_1 + \tau_2$ belong to the opposite face.

(b) For a set of four legs as above the **leg measure** is the measure supported on the legs and is equal to arc-length on the two legs $[e]$ and $[e] + \tau_1 + \tau_2$ and minus arc-length on the two legs $[e] + \tau_1$ and $[e] + \tau_2$. We denote this measure by μ_{e, τ_1, τ_2} . The leg measure is defined up to sign.

Remark 2.1. A set of four legs of a symmetric polytope with symmetric faces is determined uniquely if we know two opposite edges on a face (these edges are the first two legs). The other two legs are then the corresponding opposite edges on the opposite face.

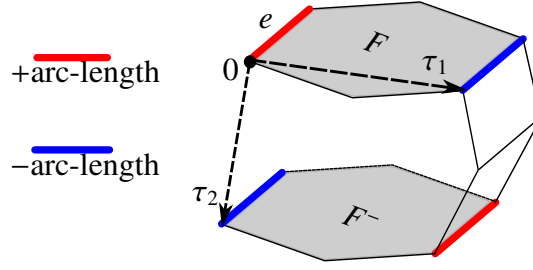


FIGURE 1. Four legs of a convex polytope and leg-measure. The colored segments are edges of the polytope. Translating an edge by $\pm\tau_1$ gives you the opposite edge on a face. Translating by $\pm\tau_2$ takes you to the opposite face.

3. THE FOURIER TRANSFORM OF A 4-LEGGED FRAME

Lemma 3.1. *Suppose $\mathbf{e}, \tau_1, \tau_2 \in \mathbb{R}^3$ are linearly independent and consider the measure $\mu = \mu_{\mathbf{e}, \tau_1, \tau_2}$ (see Definition 2.2). Then the zero-set of the Fourier Transform $\hat{\mu}$, is*

$$(4) \quad Z(\hat{\mu}) = \mathbb{H}_{-0}(\mathbf{e}) \cup \mathbb{H}(\tau_1) \cup \mathbb{H}(\tau_2),$$

where, if \mathbf{x} is a non-zero vector and $\mathbf{x}^* = \frac{\mathbf{x}}{|\mathbf{x}|^2}$ is its geometric inverse,

$$(5) \quad \mathbb{H}(\mathbf{x}) = \mathbb{Z}\mathbf{x}^* + \mathbf{x}^\perp$$

and

$$(6) \quad \mathbb{H}_{-0}(\mathbf{x}) = (\mathbb{Z} \setminus \{0\})\mathbf{x}^* + \mathbf{x}^\perp.$$

Here \mathbf{x}^\perp is the hyperplane orthogonal to the vector \mathbf{x} , so that $\mathbb{H}(\mathbf{x}) = \mathbb{Z}\mathbf{x}^* + \mathbf{x}^\perp$ is a collection of parallel hyperplanes, orthogonal to \mathbf{x} spaced by $1/|\mathbf{x}|$.

Proof. Translating a measure does not alter the zero-set of its FT so we may translate μ so that 0 is the midpoint of the first line segment, which now runs from $-\mathbf{e}/2$ to $\mathbf{e}/2$. Denoting by ν the arc-length measure on this line segment and writing $\alpha = \delta_0 - \delta_{\tau_1}$ and $\beta = \delta_0 - \delta_{\tau_2}$ we obtain μ as a convolution:

$$\mu = \nu * \alpha * \beta.$$

Taking the FT we get that

$$Z(\hat{\mu}) = Z(\hat{\nu}) \cup Z(\hat{\alpha}) \cup Z(\hat{\beta}).$$

Based on the calculation of the FT of the indicator function of $[-\frac{1}{2}, \frac{1}{2}]$

$$\int_{-1/2}^{1/2} e^{-2\pi i \xi x} dx = \frac{\sin \pi \xi}{\pi \xi}$$

we conclude that

$$\widehat{\nu}(\mathbf{u}) = |\mathbf{e}| \frac{\sin \pi \langle \mathbf{u}, \mathbf{e} \rangle}{\pi \langle \mathbf{u}, \mathbf{e} \rangle}.$$

One also immediately obtains the formulas

$$\widehat{\alpha}(\mathbf{u}) = 2ie^{-\pi i \langle \tau_1, \mathbf{u} \rangle} \sin \pi \langle \tau_1, \mathbf{u} \rangle$$

and

$$\widehat{\beta}(\mathbf{u}) = 2ie^{-\pi i \langle \tau_2, \mathbf{u} \rangle} \sin \pi \langle \tau_2, \mathbf{u} \rangle.$$

Since $\widehat{\nu}$, $\widehat{\alpha}$ and $\widehat{\beta}$ vanish precisely on $\mathbb{H}_{-0}(\mathbf{e})$, $\mathbb{H}(\tau_1)$ and $\mathbb{H}(\tau_2)$ respectively, the proof of Lemma 3.1 is complete. \square

4. A SUFFICIENT CONDITION FOR $\widehat{\delta}_\Lambda$ TO HAVE DISCRETE SUPPORT

Theorem 4.1. *Suppose P is a symmetric polytope in \mathbb{R}^3 with symmetric faces and Λ is a multiset of points in \mathbb{R}^3 such that P tiles at level k , a positive integer, when translated at the locations $\lambda \in \Lambda$. Then we have*

$$(7) \quad \text{supp } \widehat{\delta}_\Lambda \subseteq \{0\} \cup \bigcap_{e, \tau_1, \tau_2} (\mathbb{H}_{-0}(\mathbf{e}) \cup \mathbb{H}(\tau_1) \cup \mathbb{H}(\tau_2)),$$

where δ_Λ is the measure corresponding to Λ defined in (2) and the intersection in (7) is taken over all 4-legged frames (e, τ_1, τ_2) of P .

Proof. We know from [6] (see Lemma 3.1 and Lemma 3.2 in [6]) that if P tiles with Λ and μ is a leg measure on P then μ also tiles with Λ , at level 0. In other words $\mu * \delta_\Lambda = 0$. Since P tiles when translated by Λ it follows that $|\Lambda \cap [-R, R]^3| = O(R^3)$, hence δ_Λ is a tempered distribution and we may take its FT which gives us $\widehat{\mu} \widehat{\delta}_\Lambda = 0$. This implies (see the details in [12, §1.2])

$$\text{supp } \widehat{\delta}_\Lambda \subseteq \{0\} \cup Z(\widehat{\mu}).$$

But the measure μ is exactly the one described in Lemma 3.1 and since this must be true for all sets of four legs of P we conclude (7). \square

Corollary 4.1. *Suppose P is a k -tiler with a discrete multiset Λ , in \mathbb{R}^3 . Let the following intersection property hold:*

$$(8) \quad \bigcap_{e, \tau_1, \tau_2} (\mathbf{e}^\perp \cup \tau_1^\perp \cup \tau_2^\perp) = \{0\},$$

where the intersection above is taken over all sets of 4-legged frames of P .

Then $\text{supp } \widehat{\delta}_\Lambda$ is a discrete set in \mathbb{R}^3 , of bounded density.

Proof. The sets which are being intersected in (7) are all unions of planes. For this set to be non-discrete it must be the case that it contains an entire line of direction, say $\mathbf{u} \in \mathbb{R}^3 \setminus \{0\}$.

This in turn implies that there is a selection X_ℓ of \mathbf{e} , τ_1 or τ_2 for each set ℓ of four legs such that $\mathbf{u} \in X_\ell^\perp$. This contradicts condition (8).

Having established that the intersection in the right hand side of (7) is a discrete point set we observe that the larger set

$$(9) \quad \bigcap_{\mathbf{e}, \tau_1, \tau_2} \mathbb{H}(\mathbf{e}) \cup \mathbb{H}(\tau_1) \cup \mathbb{H}(\tau_2)$$

is a finite union of discrete groups, each of them of the form

$$\bigcap_{\ell} \mathbb{H}_\ell,$$

where ℓ runs through all possible sets of four legs of P and for each $\ell = \{e, \tau_1, \tau_2\}$ the set \mathbb{H}_ℓ is one of $\mathbb{H}(\mathbf{e})$, $\mathbb{H}(\tau_1)$, $\mathbb{H}(\tau_2)$. Since each discrete group has bounded density so has the set (9) and $\text{supp } \widehat{\delta}_\Lambda$ as its subset. \square

5. THE INTERSECTION PROPERTY IMPLIES QUASI-PERIODICITY OF Λ

In this section we show how the discreteness of $\text{supp } \widehat{\delta}_\Lambda$ implies a rather rigid structure for Λ . Below we quote the result from [11], where the multidimensional statements had been proved in general, despite the fact that the final conclusions in [11] are given only for dimension 2.

Theorem 5.1 (Kolountzakis, 2002). *Suppose that for the multiset $\Lambda \subseteq \mathbb{R}^d$*

- (1) Λ has uniformly bounded density;
- (2) $S := \text{supp } \widehat{\delta}_\Lambda$ is discrete;
- (3) $|S \cap B_R(0)| \leq C \cdot R^d$, for some positive constant C .

Then Λ is a finite union of translated d -dimensional lattices.

Next, we verify the conditions of the theorem above, for our 3-dimensional k -tilers with a multiset Λ .

Claim 5.1. *Suppose that convex polytope P k -tiles \mathbb{R}^d with Λ and the intersection property (8) of Corollary 4.1 is true. Then Λ is a finite union of translated d -dimensional lattices.*

Proof. We just need to verify conditions (1), (2) and (3) of theorem 5.1.

Hypothesis (1) simply follows from the fact that in each sufficiently large ball $B_R(x)$ of \mathbb{R}^d every point is covered exactly k times by the translations of P with the set $\Lambda \cap B_{R'}(x)$, where $R' = R + \text{diam } P$.

Hypotheses (2) and (3) follow from Corollary 4.1. \square

6. k -TILERS ASSOCIATED TO A NON-DISCRETE $\text{supp } \widehat{\delta}_\Lambda$

In this section we study the convex polytopes that admit exceptional multiple tilings, in the sense that the multiset of translations Λ is not a finite union of 3-dimensional lattices. A class of these exceptions is easily provided by prisms (Minkowski sums of a symmetric polygon with a line segment, not in the polygons plane), for which one can lift a 2-dimensional k -tiling up into the third dimension by separately 1-tiling the tube above each projection.

By Claim 5.1 for such a tiling, the intersection property (8) cannot be true. Therefore, there exists a line (in fact a 1-dimensional subspace) $l \subseteq \mathbb{R}^3$ such that

$$(10) \quad l \subseteq \bigcap_{\mathbf{e}, \tau_1, \tau_2} (\mathbf{e}^\perp \cup \tau_1^\perp \cup \tau_2^\perp).$$

It was already shown in [6] that a multiple tiler in \mathbb{R}^3 must be a zonotope, i. e. a Minkowski sum of line segments. Here we will show that given the non-discreteness of $\text{supp } \widehat{\delta}_\Lambda$, we can deduce that a zonotope is a Minkowski sum of two 2-dimensional symmetric polygons. And in Section 7 we provide examples of such exceptional tilings, under a mild commensurability condition for such zonotopes.

Theorem 6.1. *Suppose a polytope P tiles \mathbb{R}^3 with multiplicity by translations over a multiset Λ and condition (10) holds. Then P is a two-flat zonotope.*

Proof. We let L be a plane orthogonal to l and supporting P ; (10) is then equivalent to

$$(11) \quad \forall \mathbf{e}, \tau_1, \tau_2, \text{ either } \mathbf{e} \parallel L, \text{ or } \tau_1 \parallel L, \text{ or } \tau_2 \parallel L.$$

Let $F = L \cap P$. The dimension of the face F can be 0, 1 or 2. Consider any facet G of P that has at least one common vertex with F , and let e be an edge of G that shares exactly one vertex v with F (so $G \neq F$). Consider the 4-legged frame determined by G and e with τ_1, τ_2 being the corresponding translation vectors. Since v is in L , by (11) one of the three vertices $v + \mathbf{e}$, $v + \tau_1$ and $v + \tau_2$ lies in L , and therefore lies in F as well. By our choice of \mathbf{e} , $v + \mathbf{e}$ is a vertex of G but not of F . Thus either $v + \tau_1 \in F$, or $v + \tau_2 \in F$:

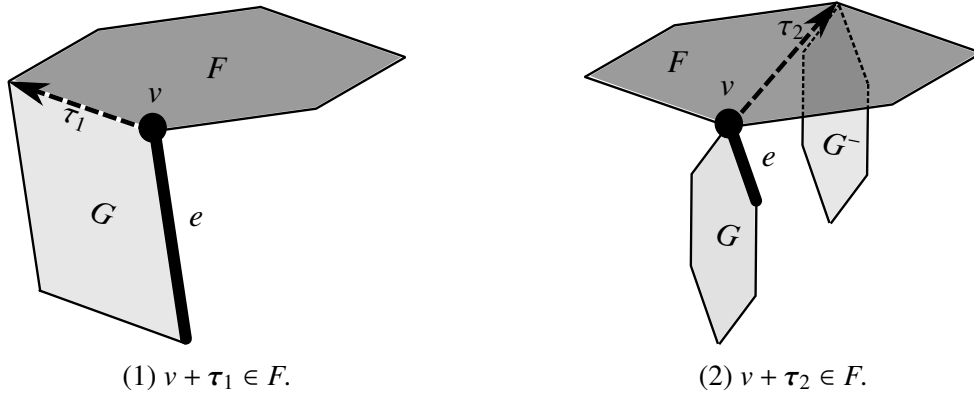
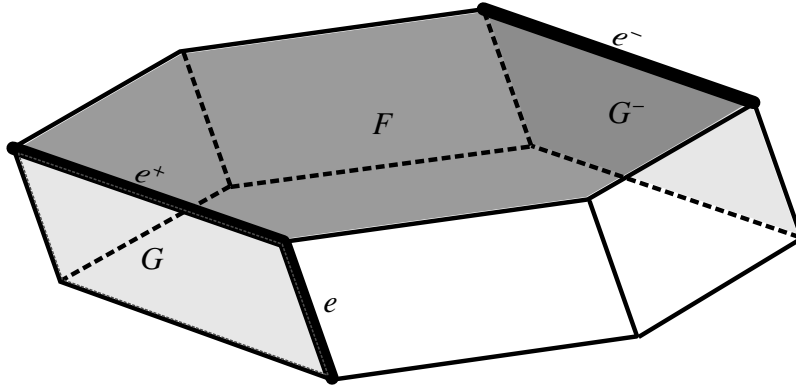
(1) If $v + \tau_1 \in F$, then $\tau_1 \in G \cap F$, so we see that τ_1 is an edge of G . Hence G is a parallelogram.

(2) If $v + \tau_2 \in F$, then F connects G with its opposite face G^- .

See Fig. 2.

It is our goal to find a facet G which satisfies property (2). If, to the contrary, every facet adjacent to F satisfies property (1), then each facet adjacent to F is a parallelogram sharing an edge with F . It follows that exactly three edges meet at every vertex of F , and all edges of these parallelograms that are not edges of F or parallel to F , are parallel to each other (see Fig. 3). Now since F is centrally symmetric, consider two opposite parallel edges e^+ and e^- of F , and corresponding parallelogram facets G and G^- . The facets G and G^- are parallel and therefore opposite in P , therefore G enjoys property (2).

Now that we have found a facet G such that F connects G with G^- , we also note that since P is centrally symmetric, G also connects F and F^- . We will now show that $P = F + G$, under the

FIGURE 2. The two possibilities for the facet G with respect to F FIGURE 3. This is the case that each facet adjacent to F is a parallelogram sharing an edge with F , giving us a prism

minor assumption that F and G do not share an edge. The case that F and G do in fact share an edge may be handled in exactly the same manner, so without loss of generality we assume throughout the rest of the proof that F and G share none of their edges. Since P is a zonotope, $P = F + G + H$ for some polytope H . To arrive at a contradiction, we assume to the contrary that H is not a single point, and let h_0 be any edge of H . Let $F = [f_1] + \dots + [f_k]$, $G = [g_1] + \dots + [g_\ell]$, and $H = [h_0] + [h_1] + \dots + [h_m]$, where $k \geq 0$, $\ell \geq 2$, and $m \geq 0$. We may assume that all line segments f_i, g_i, h_i have the origin as their midpoint and thus the center of P is also at the origin. We further consider a normal vector \mathbf{f}_\perp to the face F of P . When F is a 2-dimensional face of P , \mathbf{f}_\perp cannot be orthogonal to any line segment $h_i \in H$ and $g_i \in G$. If F is 0 or 1 dimensional face of P (see figure 4), we have an infinite collection of perpendicular vectors to F and we may choose \mathbf{f}_\perp to be not orthogonal to any line segment $h_i \in H$ and $g_i \in G$. For each edge g_i (resp. h_i) we define \mathbf{g}_i^+ (resp. \mathbf{h}_i^+) to be the vector from the origin to the endpoint of g_i (resp. h_i) such that $\langle \mathbf{g}_i^+, \mathbf{f}_\perp \rangle > 0$ (resp. $\langle \mathbf{h}_i^+, \mathbf{f}_\perp \rangle > 0$). In the same way we define \mathbf{g}_i^- (resp. \mathbf{h}_i^-) s.t. $\langle \mathbf{g}_i^-, \mathbf{f}_\perp \rangle < 0$ (resp. $\langle \mathbf{h}_i^-, \mathbf{f}_\perp \rangle < 0$). Now one may easily see that the location of the face F in \mathbb{R}^3 is given by $[f_1] + \dots + [f_k] + \mathbf{g}_1^+ + \dots + \mathbf{g}_\ell^+ + \mathbf{h}_0^+ + \dots + \mathbf{h}_m^+$ as a set of extremal points of the linear functional corresponding to \mathbf{f}_\perp . Similarly the location of the face F^- in \mathbb{R}^3 is given by $[f_1] + \dots + [f_k] + \mathbf{g}_1^- + \dots + \mathbf{g}_\ell^- + \mathbf{h}_0^- + \dots + \mathbf{h}_m^-$. Therefore the distance between F and F^- is

$$\langle \mathbf{f}_\perp, \sum_{i=1}^{\ell} \mathbf{g}_i^+ + \sum_{i=0}^m \mathbf{h}_i^+ - \sum_{i=1}^{\ell} \mathbf{g}_i^- - \sum_{i=0}^m \mathbf{h}_i^- \rangle > \langle \mathbf{f}_\perp, \sum_{i=1}^{\ell} (\mathbf{g}_i^+ - \mathbf{g}_i^-) \rangle.$$

On the other hand, since G connects F and F^- we have $F = F^- + \sum_{i \in I} (\mathbf{g}_i^+ - \mathbf{g}_i^-)$, for a set I of edges in G . The latter implies that the distance between F and F^- is not more than

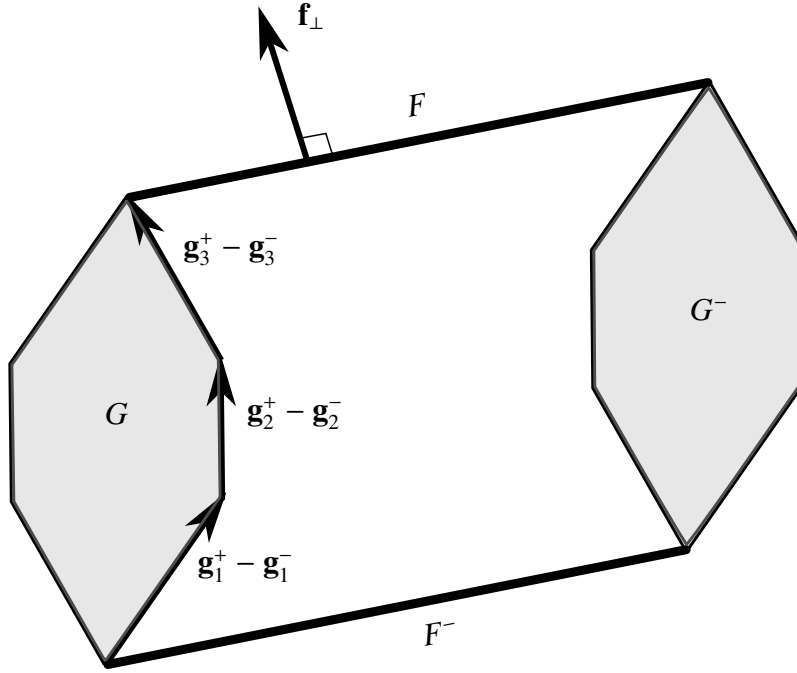


FIGURE 4. Here F is a lower-dimensional face of P , namely an edge of P , and we see how we can get from the face F to the face F^- by walking along the vectors $\mathbf{g}_i^+ - \mathbf{g}_i^-$. Here \mathbf{f}_\perp is chosen to be a vector orthogonal to F and not orthogonal to any of the line segments h_j .

$$\langle \mathbf{f}_\perp, \sum_{i=1}^{\ell} (\mathbf{g}_i^+ - \mathbf{g}_i^-) \rangle,$$

a contradiction. □

Remark 6.1. One of F or G can be 1-dimensional, in which case P becomes a 3-dimensional prism.

Main Theorem. Suppose a polytope P k -tiles \mathbb{R}^3 with a multiset Λ , and suppose that P is not a two-flat zonotope. Then Λ is a finite union of translated lattices.

Proof. If P is not a two-flat zonotope, then due to Theorem 6.1, condition (10) is violated. Therefore, the intersection property (8) in Corollary 4.1 holds. The Claim 5.1 now concludes the proof. □

7. MANY TWO-FLAT ZONOTOPES HAVE WEIRD TILING SETS

In this section we prove that, under a mild commensurability condition, two-flat zonotopes admit tilings which are not quasi-periodic (“weird”).

Theorem 7.1. Suppose P is a two-flat zonotope in \mathbb{R}^3 which is the Minkowski sum of the segments

$$[\mathbf{v}_1], \dots, [\mathbf{v}_n], [\mathbf{w}_1], \dots, [\mathbf{w}_m],$$

where $\mathbf{v}_1, \dots, \mathbf{v}_n \in H_1$ and $\mathbf{w}_1, \dots, \mathbf{w}_m \in H_2$ and H_1, H_2 are two different two dimensional subspaces. Suppose also that the additive group generated by $\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{w}_1, \dots, \mathbf{w}_m$ is discrete and that the \mathbf{v}_j span H_1 .

Then P admits a tiling by translations at a set $\Lambda \subseteq \mathbb{R}^3$ which is not a finite union of translated lattices.

Corollary 7.1. If $P \subseteq \mathbb{R}^3$ is a two-flat rational zonotope then P admits tilings by sets which are not finite unions of translated lattices.

Proof of Theorem 7.1. We begin the analysis by noting that P can be paved by parallelepipeds, whose sides are among the vectors \mathbf{v}_j and \mathbf{w}_j (proof is by induction on the number of line segments whose Minkowski sum is the zonotope). Therefore we can write its indicator function as a finite sum of indicator functions of parallelepipeds.

$$\mathbf{1}_P(x) = \sum_{j=1}^M \mathbf{1}_{B_j}(x), \quad \text{for a.e. } x,$$

where each B_j is a parallelepiped, whose three sides are equal to some of the \mathbf{v}_j and \mathbf{w}_j .

Suppose now that the parallelepiped B is centered at the origin and has sides parallel to the three linearly-independent vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$. We can write the indicator function of B as a convolution

$$\mathbf{1}_B = \frac{|\det(\mathbf{a}, \mathbf{b}, \mathbf{c})|}{|\mathbf{a}| \cdot |\mathbf{b}| \cdot |\mathbf{c}|} \mu_{\mathbf{a}} * \mu_{\mathbf{b}} * \mu_{\mathbf{c}},$$

where $\mu_{\mathbf{a}}$ is the measure that equals arc-length on the line segment from $-\mathbf{a}/2$ to $\mathbf{a}/2$, and $\mu_{\mathbf{b}}, \mu_{\mathbf{c}}$ are similarly defined. Since (see §3)

$$\widehat{\mu}_{\mathbf{a}}(\boldsymbol{\xi}) = |\mathbf{a}| \frac{\sin \pi \langle \boldsymbol{\xi}, \mathbf{a} \rangle}{\pi \langle \boldsymbol{\xi}, \mathbf{a} \rangle}$$

and similarly for $\widehat{\mu}_{\mathbf{b}}, \widehat{\mu}_{\mathbf{c}}$, we obtain the formula

$$(12) \quad \widehat{\mathbf{1}}_B(\boldsymbol{\xi}) = |\det(\mathbf{a}, \mathbf{b}, \mathbf{c})| \frac{\sin \pi \langle \boldsymbol{\xi}, \mathbf{a} \rangle}{\pi \langle \boldsymbol{\xi}, \mathbf{a} \rangle} \cdot \frac{\sin \pi \langle \boldsymbol{\xi}, \mathbf{b} \rangle}{\pi \langle \boldsymbol{\xi}, \mathbf{b} \rangle} \cdot \frac{\sin \pi \langle \boldsymbol{\xi}, \mathbf{c} \rangle}{\pi \langle \boldsymbol{\xi}, \mathbf{c} \rangle}.$$

Each parallelepiped B_j in the decomposition of P is a translate of a parallelepiped of the type B , above, with some of the vectors $\mathbf{v}_j, \mathbf{w}_j$ in place of $\mathbf{a}, \mathbf{b}, \mathbf{c}$. Hence the Fourier Transform $\widehat{\mathbf{1}}_{B_j}$ has the same zeros as the Fourier Transform of its centered translate and these are

$$\mathbf{Z}(\widehat{\mathbf{1}}_{B_j}) = ((\mathbb{Z}')\mathbf{a}^* + \mathbf{a}^\perp) \cup ((\mathbb{Z}')\mathbf{b}^* + \mathbf{b}^\perp) \cup ((\mathbb{Z}')\mathbf{c}^* + \mathbf{c}^\perp),$$

where $\mathbb{Z}' = \mathbb{Z} \setminus \{0\}$ and again $\mathbf{a}^* = \mathbf{a}/|\mathbf{a}|^2$ is the geometric inverse of \mathbf{a} , etc. Write now

$$G = \langle \mathbf{v}_1, \dots, \mathbf{v}_n \rangle$$

for the additive subgroup (lattice) of H_1 generated by the \mathbf{v}_j 's and $G^* \subseteq H_1$ for its dual lattice in H_1

$$(13) \quad G^* = \{\mathbf{u} \in H_1 : \forall \mathbf{g} \in G \quad \langle \mathbf{u}, \mathbf{g} \rangle \in \mathbb{Z}\}.$$

We claim now that for each j

$$(14) \quad H_1^\perp + (G^* \setminus (\mathbf{v}_1^\perp \cup \dots \cup \mathbf{v}_n^\perp)) \subseteq \mathbf{Z}(\widehat{\mathbf{1}}_{B_j}).$$

This follows since at least one side of B_j is equal to a vector \mathbf{v}_j which makes the corresponding factor in (12) vanish on any element of G^* which is not orthogonal to \mathbf{v}_j . And since that factor in (12) is constant along H_1^\perp we obtain the claim. Since (14) holds for all j we obtain

$$(15) \quad H_1^\perp + (G^* \setminus (\mathbf{v}_1^\perp \cup \dots \cup \mathbf{v}_n^\perp)) \subseteq \mathbf{Z}(\widehat{\mathbf{1}}_P).$$

Pick now any non-zero $c_1, c_2, \dots, c_n \in \mathbb{R}$. We claim that the measure

$$(16) \quad \tau := \mathbf{1}_P * \delta_G * [\delta_0 - \delta_{c_1 \mathbf{v}_1}] * \dots * [\delta_0 - \delta_{c_n \mathbf{v}_n}] = 0,$$

where $\delta_G = \sum_{\mathbf{g} \in G} \delta_{\mathbf{g}}$. For this it is enough to show that the Fourier Transform of the above measure

$$\widehat{\tau}(\boldsymbol{\xi}) = \widehat{\mathbf{1}}_P(\boldsymbol{\xi}) (1 - e^{2\pi i c_1 \langle \mathbf{v}_1, \boldsymbol{\xi} \rangle}) \dots (1 - e^{2\pi i c_n \langle \mathbf{v}_n, \boldsymbol{\xi} \rangle}) \widehat{\delta}_G$$

is identically 0. By the Poisson Summation Formula (the Fourier Transform is taken in the sense of distributions)

$$(17) \quad \widehat{\delta}_L = \frac{1}{\text{vol } L} \delta_{L^*},$$

for each lattice $L = AZ^d$ in \mathbb{R}^d and dual lattice $L^* = A^{-T}\mathbb{R}^d$ (where $A \in GL(d, \mathbb{R})$), it follows that $\widehat{\delta}_G$ is a *measure* with support on the lines orthogonal to H_1 that go through the points of G^* :

$$\text{supp } \widehat{\delta}_G = G^* + H_1^\perp.$$

By (15) the function $\widehat{\mathbf{1}}_P(\xi)$ kills $\widehat{\delta}_G$ except at the lines of the form $\mathbf{g}^* + H_1^\perp$ with $\mathbf{g}^* \in G^*$ is orthogonal to some \mathbf{v}_j . But at these lines one of the factors

$$(1 - e^{2\pi i c_1 \langle \mathbf{v}_1, \xi \rangle}) \dots (1 - e^{2\pi i c_n \langle \mathbf{v}_n, \xi \rangle})$$

vanishes, so indeed $\widehat{\tau}$ is zero. Now rewrite the measure $(\delta_0 - \delta_{c_1 \mathbf{v}_1}) \dots (\delta_0 - \delta_{c_n \mathbf{v}_n})$ in the form

$$\sum_{k=1}^N \delta_{\mathbf{u}_k^+} - \sum_{k=1}^N \delta_{\mathbf{u}_k^-}, \quad (\text{where } N = 2^{n-1}).$$

Equivalently, we can rewrite (16) as the equality

$$(18) \quad \mathbf{1}_P * \delta_G * \sum_{k=1}^N \delta_{\mathbf{u}_k^+} = \mathbf{1}_P * \delta_G * \sum_{k=1}^N \delta_{\mathbf{u}_k^-}.$$

Define the multisets

$$(19) \quad S = G + \{\mathbf{u}_1^+, \dots, \mathbf{u}_N^+\} \text{ and } T = G + \{\mathbf{u}_1^-, \dots, \mathbf{u}_N^-\}$$

whose ground sets are the supports of the discrete measures

$$\delta_G * \sum_{k=1}^N \delta_{\mathbf{u}_k^+} \text{ and } \delta_G * \sum_{k=1}^N \delta_{\mathbf{u}_k^-},$$

and their multiplicities at each point are those described by these measures.

In what follows we exploit (18) to give an example of a multiple tiling by P with a **discrete set** Λ , which by no means can be expressed as a finite union of translated lattices.

We notice first that since P is a zonotope decomposing into parallelepipeds of sides among the vectors $\mathbf{v}_j, \mathbf{w}_j$, it k -tiles \mathbb{R}^3 , for some k , with the lattice

$$\Gamma = \langle \mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{w}_1, \dots, \mathbf{w}_m \rangle$$

generated by the $\mathbf{v}_j, \mathbf{w}_j$. The reason is that each of the parallelepipeds B_j tiles with a subgroup of Γ (the group generated by its side vectors) and therefore it tiles multiply with Γ itself. Clearly, P also (Nk) -tiles \mathbb{R}^3 by the union of N translations of the lattice Γ by the vectors $\mathbf{u}_1^+, \dots, \mathbf{u}_N^+$.

Let $\{\gamma_j : j \in \mathbb{Z}\}$ be a complete set of coset representatives of G in Γ . Define the set of translates

$$\Lambda = \bigcup_{j \in \mathbb{Z}} (E_j + \gamma_j),$$

where for each $j \in \mathbb{Z}$ we choose $E_j = S$ or $E_j = T$ arbitrarily. We claim that for any such choice of the E_j the Λ -translates of P form a (Nk) -tiling of \mathbb{R}^3 . Indeed the claim is true if all $E_j = S$ as it is a restatement of the fact that P (Nk) -tiles with $\Gamma + \{\mathbf{u}_1^+, \dots, \mathbf{u}_N^+\}$, which itself follows from the fact that P k -tiles with Γ . Observe now that if we change any single E_j from S to T we are adding the quantity

$$(20) \quad \mathbf{1}_P * \delta_G * \sum_{i=1}^N \delta_{\mathbf{u}_i^-} * \delta_{\gamma_j} - \mathbf{1}_P * \delta_G * \sum_{i=1}^N \delta_{\mathbf{u}_i^+} * \delta_{\gamma_j}$$

to the constant function

$$\mathbf{1}_P * \delta_\Lambda,$$

which therefore remains the same since (20) is identically 0 by (18). We conclude that we have a (Nk) -tiling no matter how the E_j are chosen (one has to make the remark here that in any given bounded region of space the fact that $P + \Lambda$ is a (Nk) -tiling or not is affected by finitely many choices for the E_j).

Choose now all $E_j = S$ with the exception of $E_0 = T$. We claim that the corresponding set Λ is not a finite union of translated fully-dimensional lattices. Indeed, by the Poisson Summation Formula (17) we have that, if

$$\Lambda' = \bigcup_{j \in \mathbb{Z}} (S + \gamma_j) = \Gamma + \{\mathbf{u}_1^+, \dots, \mathbf{u}_N^+\}$$

then $\widehat{\delta_{\Lambda'}}$ is a discrete measure in \mathbb{R}^3 and this should also be true for $\widehat{\delta_{\Lambda}}$ if Λ too were a finite union of translated lattices. Thus the difference

$$\widehat{\delta_{\Lambda'}} - \widehat{\delta_{\Lambda}}$$

would also be a discrete measure. But

$$\begin{aligned} \delta_{\Lambda'} - \delta_{\Lambda} &= \delta_{S+\gamma_0} - \delta_{T+\gamma_0} \\ &= \delta_{\gamma_0} * \delta_G * \sum_{i=1}^N (\delta_{\mathbf{u}_i^+} - \delta_{\mathbf{u}_i^-}) \\ &= \delta_{\gamma_0} * \delta_G * (\delta_0 - \delta_{c_1 \mathbf{v}_1}) * \dots * (\delta_0 - \delta_{c_n \mathbf{v}_n}). \end{aligned}$$

so

$$(21) \quad \widehat{\delta_{\Lambda'}} - \widehat{\delta_{\Lambda}} = e^{2\pi i \langle \gamma_0, \xi \rangle} \prod_{j=1}^n (1 - e^{2\pi i c_j \langle \mathbf{v}_j, \xi \rangle}) \widehat{\delta_G}.$$

Recall now that the support of the measure $\widehat{\delta_G}$ are all straight lines orthogonal to H_1 passing through a point of G^* , the dual lattice to G in H_1 . The factors in the right hand side of (21) vanish at the set

$$(22) \quad \bigcup_{j=1}^n \left(\mathbb{Z} \frac{\mathbf{v}_j^*}{c_j} + \mathbf{v}_j^\perp \right).$$

Each set in this union consists of a series of planes normal to \mathbf{v}_j and spaced by a length of $(c_j |\mathbf{v}_j|)^{-1}$. Each of the straight lines that make up the support of $\widehat{\delta_G}$ is parallel to each such plane and, therefore, each such line is either entirely contained in (22) or is disjoint from it. It follows that, since the right-hand side of (21) is not identically zero, its support contains at least one straight line of the direction H_1^\perp and is not a discrete set, as we had to show. \square

Remark 7.1. *One may easily extend the previous construction of Λ to the examples that cannot be expressed as a linear combination of finitely many possibly lower-dimensional lattices.*

Proof. In the previous construction of Λ we could let E_j to be either S , or T for each j and still get a legitimate (Nk) -tiling of \mathbb{R}^3 . In general we could have a big family (of cardinality $2^{\mathbb{Z}}$) of possible Nk -tilings of \mathbb{R}^3 . We call a tiling **weird** if it is not quasi-periodic. In what follows we show that our big family has a weird member Λ^\dagger . In our construction we will need the following claim.

Claim 7.1. *The set of integers \mathbb{Z} can be colored with two colors in such a way that every arithmetic progression has infinitely many numbers of each color.*

Proof. There are countably many arithmetic progressions in \mathbb{Z} . We enumerate them all denoting A_i the the i 'th progression in the enumeration, such that any progression appears infinitely many times. We begin to color \mathbb{Z} step by step in such a way, that on the n 'th step all progressions A_i for i from 1 to n have numbers of both colors. At the n 'th step we find two numbers of A_n that are not yet colored, and color them differently. It is always possible to do so, because at step n only finitely many numbers of \mathbb{Z} are already colored, and A_n has infinitely many numbers. With such a coloring every arithmetic progression would contain infinitely many integers of each of the colors. \square

We can pick our complete set of the coset representatives $\{\gamma_j : j \in \mathbb{Z}\}$ so that it contains $\gamma_1 \cdot \mathbb{Z}$ as a subset.

In order to construct Λ^\dagger , we consider a coloring of $\gamma_1 \cdot \mathbb{Z}$ with two colors (red and black) so that every arithmetic progression there has infinitely many red and infinitely many black members. We let $E_j = S$ if corresponding coset representative $\gamma_j \notin \gamma_1 \cdot \mathbb{Z}$. For coset representatives in $\gamma_1 \cdot \mathbb{Z}$, if the point $\gamma_1 \cdot j$ is red we choose $E_j = S$, if the point $\gamma_1 \cdot j$ is black we choose $E_j = T$ in Λ^\dagger .

We further notice that due to the freedom to choose c_k 's in the definition of \mathbf{u}_k^- 's and \mathbf{u}_k^+ 's, one can pick c_k 's so that multisets S and T have different multiplicities at 0. Indeed, we may pick c_k 's so that for any set of indexes $I \subset [n]$ the corresponding linear combination $\sum_{k \in I} c_k \cdot \mathbf{v}_k \notin G$. Then $G + \sum_{k=1}^N \mathbf{u}_k^+$ has 0 at multiplicity 1, while $G + \sum_{k=1}^N \mathbf{u}_k^-$ has 0 at multiplicity 0. Furthermore, at each point $\gamma_1 \cdot \ell$ of $\gamma_1 \cdot \mathbb{Z}$ the multisets $S + \ell \cdot \gamma_1$ and $T + \ell \cdot \gamma_1$ have different multiplicities as well. Thus we get irregular behavior of Λ^\dagger on the line $\mathbb{Z} \cdot \gamma_1$. In particular, Λ^\dagger simultaneously contains and misses infinitely many members of each infinite coset of $\mathbb{Z} \cdot \gamma_1$.

Now if we assume that Λ^\dagger may be expressed as a finite linear combination of translated lattices $\delta_{\Lambda^\dagger} = q_1 \cdot \delta_{\Lambda_1} + \dots + q_m \cdot \delta_{\Lambda_m}$ (to simplify notations, we will write $q_1 \cdot \Lambda_1 + \dots + q_m \cdot \Lambda_m$ instead of δ_{Λ^\dagger}), then

$$\Lambda^\dagger \cap \mathbb{Z} \cdot \gamma_1 = \sum_{i=1}^m q_i \cdot (\Lambda_i \cap \mathbb{Z} \cdot \gamma_1).$$

Each $\Lambda_i \cap \mathbb{Z} \cdot \gamma_1$ is a coset of $\mathbb{Z} \cdot \gamma_1$. Therefore, $\Lambda_i \cap \mathbb{Z} \cdot \gamma_1$ is either empty, or is a single point, or is an arithmetic progression in $\mathbb{Z} \cdot \gamma_1$ with the common difference d_i . We denote the set of all the indices of the latter Λ_i 's by $M \subset \{1, \dots, m\}$. We further consider an arithmetic progression A of $\mathbb{Z} \cdot \gamma_1$ with the common difference $\prod_{i \in M} d_i$. We notice that for any $i \in M$ either $A \cap \Lambda_i = A$, or $A \cap \Lambda_i = \emptyset$. Since $A \subset \mathbb{Z} \cdot \gamma_1$, we have

$$\Lambda^\dagger \cap A = \sum_{i=1}^m q_i \cdot (\Lambda_i \cap A) = \sum_{i \notin M} q_i \cdot (\Lambda_i \cap A) + A \cdot \sum_{\substack{i \in M: \\ \Lambda_i \cap A \neq \emptyset}} q_i.$$

According to the definition of M the set $\sum_{i \notin M} q_i \cdot \delta_{\Lambda_i \cap A}$ has finite support. Since A is an arithmetic progression in $\mathbb{Z} \cdot \gamma_1$ and due to our construction of Λ^\dagger , the support of

$$\delta_{\Lambda^\dagger \cap A} - \delta_A \cdot \sum_{\substack{i \in M: \\ \Lambda_i \cap A \neq \emptyset}} q_i$$

cannot be finite, a contradiction. □

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