

TILINGS, PACKINGS, AND THE EXISTENCE OF SCHWARTZ-CLASS GABOR WINDOWS

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ABSTRACT. The existence and construction of Schwartz-class windows for lattice Gabor frames is a central problem in time–frequency analysis. For general time–frequency lattices, we characterize exactly those lattices that admit Schwartz-class windows and, additionally, windows in the Feichtinger algebra. Thereby we arrive at a sharp Amalgam Balian–Low theorem for lattices in $\mathbb{R}^d$. For separable lattices and specified block-zero forms, we construct smooth windows that are compactly supported either in time or in frequency. These constructions are based on a characterization of pairs of lattices for which there exists a single set that tiles with one lattice and whose ε -neighborhood packs with the other.

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1. INTRODUCTION AND MAIN RESULTS

The search for measurable sets that tile Euclidean space by translations from two different lattices, or that tile by one lattice and pack by another, is a well-studied problem at the intersection of harmonic analysis and discrete geometry. In this paper we sharpen the second question. We ask not merely that the packing translates be disjoint, but that

distinct packing translates remain a positive Euclidean distance apart; equivalently, a positive Euclidean neighborhood of the tile must still pack by the second lattice.

Gabor frames, on the other hand, play a central role in time–frequency analysis and signal processing because they provide stable expansions into translated and modulated copies of a window function. A long-standing problem is to decide, for a prescribed time–frequency lattice, whether one can choose a frame window with strong localization and regularity. We settle this question for Schwartz class windows by characterizing exactly the phase-space lattices that admit a Gabor frame generated by such a window.

The two problems are intimately connected. For a separable phase-space lattice, the tiling–packing criterion for a pair of Euclidean lattices is exactly the arithmetic condition that governs the existence of a Schwartz class Gabor frame. Moreover, a positively separated tiling–packing configuration can be mollified and normalized to produce a compactly supported smooth Parseval Gabor window. Thus the geometric theorem supplies both the intuition and the constructive core of the Gabor-frame existence result.

A *full-rank lattice* in $\mathbb{R}^d$ is a subgroup of the form $L = M\mathbb{Z}^d$ with $M \in GL_d(\mathbb{R})$. A measurable set $F \subset \mathbb{R}^d$ is a *fundamental domain* for L if the translates $\{F + \ell : \ell \in L\}$ form a pairwise disjoint partition of $\mathbb{R}^d$; equivalently, every $x \in \mathbb{R}^d$ has a unique representation $x = f + \ell$ with $f \in F$ and $\ell \in L$. The *covolume* and *density* of L are

$$\text{covol}(L) = |\det M| = |F|, \quad \text{density}(L) = \frac{1}{\text{covol}(L)} = \frac{1}{|F|}.$$

Thus the covolume is the measure of a fundamental domain and the density is its reciprocal. Throughout, $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{R}^d$.

1.1. Simultaneous tilings and ε -packings

Let $T, P \subset \mathbb{R}^d$ be full-rank lattices, and let $\Omega \subset \mathbb{R}^d$ be a bounded Borel T -fundamental domain. For $\varepsilon > 0$, set

$$B_\varepsilon = \{x \in \mathbb{R}^d : \|x\|_2 < \varepsilon\}, \quad \Omega_\varepsilon = \Omega + B_\varepsilon.$$

We say that Ω_ε *packs with* P if

$$(\Omega_\varepsilon + p) \cap \Omega_\varepsilon = \emptyset, \quad p \in P \setminus \{0\}.$$

The first problem of the paper is to determine exactly when such a pair (Ω, ε) exists.

The closure of $T + P$ is a closed subgroup of $\mathbb{R}^d$ containing a full-rank lattice. Hence there is a unique $k \in \{0, \dots, d\}$ such that

$$\overline{T + P} \cong \mathbb{Z}^k \times \mathbb{R}^{d-k}.$$

We call k the *commensurability rank* of (T, P) . The case $k = d$ is exactly the commensurable case, in which $T + P$ is a lattice.

Theorem 1.1. *Let $T, P \subset \mathbb{R}^d$ be full-rank lattices. The following are equivalent.*

- (a) *There exist a bounded Borel T -fundamental domain Ω and $\varepsilon > 0$ such that $\Omega + B_\varepsilon$ packs with P .*
- (b) *The condition (TeP) holds, namely,*

$$\text{covol}(T) < \text{covol}(P) \quad \text{and, if } T + P \text{ is discrete,} \quad [T + P : P] \geq [T + P : T] + d. \quad (\text{TeP})$$

Whenever these conditions hold, Ω may be chosen as a finite union of bounded half-open polyhedra (polygons when $d = 2$).

The proof yields a stronger functional formulation. Whenever (TeP) holds, there exist both a nonnegative compactly supported continuous piecewise-affine T -partition of unity

φ_{pa} and a nonnegative compactly supported smooth T -partition of unity φ_∞ . Each may be chosen with a positive support buffer: for suitable $\eta_{\text{pa}}, \eta_\infty > 0$,

$$(\text{supp } \varphi_{\text{pa}} + B_{\eta_{\text{pa}}}) + P \quad \text{and} \quad (\text{supp } \varphi_\infty + B_{\eta_\infty}) + P$$

are packings. This is stated precisely in Corollary 9.6.

The polyhedral conclusion is obtained in three slightly different ways. In the discrete case, Section 5 constructs the required compactly supported piecewise-affine partition of unity directly. If $T + P$ is dense in $\mathbb{R}^d$, Section 8 gives a direct piecewise-affine partition-of-unity construction. In the remaining non-discrete cases, the partition-of-unity/slot argument first produces a continuous compactly supported partition of unity, and Lemma 9.5 replaces it by piecewise-affine and smooth partitions of unity without changing the partition-of-unity identity or the packing buffer.

The strict covolume inequality in (TeP) is necessary by volume comparison: if such an Ω and ε exist, then $\text{covol}(T) = |\Omega| < |\Omega + B_\varepsilon| \leq \text{covol}(P)$.

Examples 1.2. In dimension $d = 2$, the criterion (TeP) gives the following.

- (i) Let $T = \mathbb{Z}^2$ and $P = \mathbb{Z} \times \sqrt{2}\mathbb{Z}$. Then $T + P = \mathbb{Z} \times (\mathbb{Z} + \sqrt{2}\mathbb{Z})$ is not discrete and $\text{covol}(T) = 1 < \sqrt{2} = \text{covol}(P)$, so (TeP) holds.
- (ii) Let $T = \mathbb{Z}^2$ and $P = \mathbb{Z} \times 2\mathbb{Z}$. Here $T + P = \mathbb{Z}^2$, $[T + P : P] = 2$, and $[T + P : T] = 1$; hence $2 < 1 + 2$ and (TeP) fails.
- (iii) Let $T = 7\mathbb{Z} \times \mathbb{Z}$ and $P = 3\mathbb{Z} \times 3\mathbb{Z}$. Here $T + P = \mathbb{Z}^2$, $[T + P : P] = 9$, and $[T + P : T] = 7$; thus $9 = 7 + 2$ and (TeP) holds at equality.
- (iv) Let $K = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, $T = \mathbb{Z}^2$, and $P = K\mathbb{Z}^2$. Then P is non-diagonal, $T + P = \mathbb{Z}^2$, $[T + P : P] = 4$, and $[T + P : T] = 1$, so $4 \geq 1 + 2$ and (TeP) holds.

The two cases in (TeP), according to whether $T + P$ is discrete or not, are proved by closely related constructions. In the commensurable case, a finite row–column table leads to a soft configuration space, and an elementary sphere-filling argument supplies the continuously varying choices. When $T + P$ is dense in all of $\mathbb{R}^d$, there is no residual parameter torus, and a direct piecewise-affine partition-of-unity assignment suffices. In the intermediate non-discrete case, the connected directions are refined into compactly supported elements of a partition of unity and separated target slots; the resulting larger finite table is handled by the same sphere-filling argument over a lower-dimensional torus and then by the regularity replacement lemma of Section 9.6.

1.2. Characterization of lattices with Schwartz class Gabor windows

For $x, \xi, t \in \mathbb{R}^d$, the *time shift*, *frequency shift*, and *time–frequency shift* are

$$(T_x f)(t) = f(t - x), \quad (M_\xi f)(t) = e^{2\pi i \langle \xi, t \rangle} f(t), \quad \pi(x, \xi) = M_\xi T_x.$$

A family $(f_j)_{j \in J} \subset L^2(\mathbb{R}^d)$ is a *frame* if there are constants $0 < A \leq B < \infty$ such that

$$A\|f\|_2^2 \leq \sum_{j \in J} |\langle f, f_j \rangle|^2 \leq B\|f\|_2^2, \quad f \in L^2(\mathbb{R}^d).$$

It is *tight* if $A = B$ and *Parseval* if $A = B = 1$. A full-rank lattice $\Lambda \subset \mathbb{R}^{2d}$ indexes the *Gabor system*

$$\mathcal{G}(g, \Lambda) = \{\pi(\lambda)g : \lambda \in \Lambda\};$$

when this family is a frame, it is called a *Gabor frame*. Standard background on Gabor systems, frame operators, dual windows, and lattice density may be found in [25, Chapters 5–7].

For a full-rank Euclidean lattice $L \subset \mathbb{R}^d$, its dual lattice is

$$L^* = \{\xi \in \mathbb{R}^d : \langle \xi, \ell \rangle \in \mathbb{Z} \text{ for every } \ell \in L\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product on $\mathbb{R}^d$. At times we shall also write $\langle x, \xi \rangle = x \cdot \xi$. If $L = M\mathbb{Z}^d$, then $L^* = M^{-T}\mathbb{Z}^d$ and $\text{covol}(L^*) = \text{covol}(L)^{-1}$.

On $\mathbb{R}^{2d} = \mathbb{R}^d \times \mathbb{R}^d$, write

$$\sigma((x, \xi), (y, \eta)) = \langle \eta, x \rangle - \langle \xi, y \rangle.$$

Equivalently, $\sigma(z, w) = z^T J w$ with

$$J = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}.$$

The adjoint lattice of Λ is

$$\Lambda^\circ = \{z \in \mathbb{R}^{2d} : \sigma(z, \lambda) \in \mathbb{Z} \text{ for all } \lambda \in \Lambda\}.$$

The integral symplectic subgroup is

$$\Lambda_{\text{int}} = \Lambda \cap \Lambda^\circ.$$

We call Λ *symplectically rational* if $[\Lambda : \Lambda_{\text{int}}] < \infty$. Equivalently, if $\Lambda = M\mathbb{Z}^{2d}$, then the symplectic Gram matrix $M^T J M$ is entrywise rational; the basis matrix M itself need not have rational entries. For a symplectically rational lattice, set

$$v(\Lambda) = [\Lambda : \Lambda_{\text{int}}]^{1/2}.$$

This is always a positive integer. Indeed, the finite group $\Lambda/\Lambda_{\text{int}}$ carries the nondegenerate alternating pairing induced by σ modulo $\mathbb{Z}$. The skew Smith normal form used in Section 13 decomposes this group as a direct sum of pairs

$$(\mathbb{Z}/q_1\mathbb{Z})^2 \oplus \cdots \oplus (\mathbb{Z}/q_s\mathbb{Z})^2.$$

Consequently $[\Lambda : \Lambda_{\text{int}}] = (q_1 \cdots q_s)^2$ and $v(\Lambda) = q_1 \cdots q_s \in \mathbb{Z}_{\geq 1}$.

Since a tight frame is converted into a Parseval frame by multiplying its window by one positive scalar, tight-frame existence and Parseval-frame existence are equivalent. We use the normalized Parseval formulation.

Theorem 1.3. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a full-rank phase-space lattice. There exists $g \in \mathcal{S}(\mathbb{R}^d)$ such that $\mathcal{G}(g, \Lambda)$ is a Parseval Gabor frame if and only if*

$$\text{covol}(\Lambda) < 1 \quad \text{and, if } \Lambda \text{ is symplectically rational,} \quad v(\Lambda) \geq v(\Lambda^\circ) + d. \quad (\text{SG})$$

Enstad–Thiel–Vilalta proved the sufficiency of (SG) [16]. Jakobsen–Luef had already covered the non-symplectically-rational case [32]; the necessity is new. In fact, we prove the following sharp lattice amalgam Balian–Low theorem. Recall briefly that $S_0(\mathbb{R}^d) = M^1(\mathbb{R}^d)$ is the Feichtinger algebra and that it is the Wiener amalgam space $W(\mathcal{FL}^1, L^1)(\mathbb{R}^d)$; a detailed definition is given below.

Theorem 1.4. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a full-rank phase-space lattice, and suppose that $\mathcal{G}(g, \Lambda)$ is a Gabor frame. If $\text{covol}(\Lambda) \geq 1$, or if Λ is symplectically rational and $v(\Lambda) < v(\Lambda^\circ) + d$, then $g \notin S_0(\mathbb{R}^d)$.*

Examples 1.5. Use the coordinate order $(x_1, x_2, \omega_1, \omega_2)$ and write $\Lambda_j = M_j\mathbb{Z}^4$. The first four matrices correspond to the four pairs in Examples 1.2, with $\Gamma = T$ and $\Phi = P^*$.

- (i) For $M_1 = I_3 \oplus (1/\sqrt{2})$, the lattice is not symplectically rational and $\text{covol}(\Lambda_1) = 1/\sqrt{2} < 1$, so (SG) holds.
- (ii) For $M_2 = I_3 \oplus (1/2)$, one has $\text{covol}(\Lambda_2) = 1/2$, $v(\Lambda_2) = 2$, and $v(\Lambda_2^\circ) = 1$; hence $2 < 1 + 2$ and (SG) fails.
- (iii) For $M_3 = \text{diag}(7, 1, 1/3, 1/3)$, one has $\text{covol}(\Lambda_3) = 7/9$, $v(\Lambda_3) = 9$, and $v(\Lambda_3^\circ) = 7$; thus (SG) holds at equality.
- (iv) Let $K = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ and $M_4 = \text{diag}(I_2, K^{-T})$, where $K^{-T} = \begin{pmatrix} 1/2 & 0 \\ -1/4 & 1/2 \end{pmatrix}$. Then $\text{covol}(\Lambda_4) = 1/4$, $v(\Lambda_4) = 4$, and $v(\Lambda_4^\circ) = 1$, so (SG) holds.

- (v) The nonseparable shear $M_5 = \begin{pmatrix} I_2 & 0 \\ I_2 & I_2 \end{pmatrix} \begin{pmatrix} (1/3)I_2 & 0 \\ 0 & I_2 \end{pmatrix} = \begin{pmatrix} (1/3)I_2 & 0 \\ (1/3)I_2 & I_2 \end{pmatrix}$ has $\text{covol}(\Lambda_5) = 1/9$, $\nu(\Lambda_5) = 9$, and $\nu(\Lambda_5^\circ) = 1$, so (SG) holds.
- (vi) Put $D_0 = \text{diag}(1, 1/2)$ and $M_6 = 2^{-1/2} \begin{pmatrix} I_2 & I_2 \\ -I_2 & I_2 \end{pmatrix} \begin{pmatrix} D_0 & 0 \\ 0 & I_2 \end{pmatrix}$. This nonseparable symplectic rotation has $\text{covol}(\Lambda_6) = 1/2$, $\nu(\Lambda_6) = 2$, and $\nu(\Lambda_6^\circ) = 1$, so (SG) fails.

For a symplectically rational lattice, the normal-form calculation below gives

$$\text{covol}(\Lambda) = \frac{\nu(\Lambda^\circ)}{\nu(\Lambda)}. \quad (1.1)$$

Consequently,

$$\text{covol}(\Lambda) < 1 \iff \nu(\Lambda) > \nu(\Lambda^\circ) \iff \nu(\Lambda) \geq \nu(\Lambda^\circ) + 1.$$

Thus the algebraic gap in (SG) already implies the strict density inequality in the symplectically rational case.

A Parseval frame gives a painless nonorthogonal expansion: for every $f \in L^2(\mathbb{R}^d)$,

$$f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)g,$$

with unconditional convergence in $L^2(\mathbb{R}^d)$. Thus reconstruction uses the same window for analysis and synthesis and no separate dual window has to be computed. This is the main reason for seeking Parseval frames when well-localized orthonormal bases are not available; see the Balian–Low discussion in Subsection 2.2. When f has additional regularity and decay, the speed of convergence of natural finite truncations is governed by the decay of the Gabor coefficients. The time–frequency localization of the window, in particular the decay of both g and $\widehat{g}$, plays a central role in that convergence; compare the painless-expansion framework of [12] and the general treatment of Gabor expansions in [25, Chapters 5–7].

For separable lattices, the geometric condition (TeP) and the Gabor-frame condition (SG) are the same arithmetic requirement. This connection is rooted in the use of dual lattices and fundamental domains in Gabor analysis; see [30, 41] and [25, Chapters 7–8].

Proposition 1.6. *Let $\Lambda = \Gamma \times \Phi$, where $\Gamma, \Phi \subset \mathbb{R}^d$ are full-rank lattices. Then*

$$\Lambda \text{ satisfies (SG)} \iff (\Gamma, \Phi^*) \text{ satisfies (TeP)}.$$

If $\Gamma + \Phi^$ is not discrete, the common condition is $\text{covol}(\Gamma) < \text{covol}(\Phi^*)$, equivalently $\text{covol}(\Lambda) < 1$. If $\Gamma + \Phi^*$ is a lattice, then*

$$\nu(\Lambda) = [\Gamma + \Phi^* : \Phi^*], \quad \nu(\Lambda^\circ) = [\Gamma + \Phi^* : \Gamma],$$

and the common condition is

$$[\Gamma + \Phi^* : \Phi^*] \geq [\Gamma + \Phi^* : \Gamma] + d.$$

At the level of arbitrary square-integrable windows, there is no obstruction beyond the ordinary density condition. Bekka proved that a full-rank phase-space lattice Λ admits a window $g \in L^2(\mathbb{R}^d)$ for which $\mathcal{G}(g, \Lambda)$ is a Gabor frame if and only if

$$\text{covol}(\Lambda) \leq 1;$$

see [5, Theorem 4]. Applying the inverse square root of the frame operator then gives an L^2 Parseval window on the same lattice. Thus every lattice satisfying the necessary density inequality admits an L^2 Gabor frame; the purpose of (SG) is to characterize when the window can be required to have strong time–frequency regularity.

We write $C_c^\infty(\mathbb{R}^d)$ for the space of infinitely differentiable functions on $\mathbb{R}^d$ with compact support, and $\mathcal{S}(\mathbb{R}^d)$ for the Schwartz class. The classical Wiener amalgam space $W(C, \ell^1)(\mathbb{R}^d)$ consists of the continuous functions g for which

$$\|g\|_{W(C, \ell^1)} := \sum_{k \in \mathbb{Z}^d} \sup_{x \in [0,1]^d} |g(x+k)| < \infty.$$

We also use the Fourier-invariant Wiener space

$$W_0(\mathbb{R}^d) := \{g \in W(C, \ell^1)(\mathbb{R}^d) : \widehat{g} \in W(C, \ell^1)(\mathbb{R}^d)\}.$$

We write $C_Z(\mathbb{R}^d)$ for the set of all $g \in L^2(\mathbb{R}^d)$ whose Zak transform has a continuous representative on $\mathbb{R}^d \times \mathbb{R}^d$. Finally, $S_0(\mathbb{R}^d) = M^1(\mathbb{R}^d)$ denotes the Feichtinger algebra. To write out its Wiener-amalgam realization, choose a nonnegative function $\chi \in C_c^\infty(\mathbb{R}^d)$ such that

$$\sum_{k \in \mathbb{Z}^d} \chi(x-k) = 1, \quad x \in \mathbb{R}^d.$$

Then

$$W(\mathcal{F}L^1, L^1)(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \sum_{k \in \mathbb{Z}^d} \left\| \mathcal{F}(f \chi(\cdot - k)) \right\|_{L^1(\mathbb{R}^d)} < \infty \right\}. \quad (1.2)$$

Different choices of such a bounded uniform partition of unity give equivalent norms. One has

$$S_0(\mathbb{R}^d) = M^1(\mathbb{R}^d) = W(\mathcal{F}L^1, L^1)(\mathbb{R}^d).$$

Equivalently, for one, and then every, nonzero $\phi \in \mathcal{S}(\mathbb{R}^d)$,

$$S_0(\mathbb{R}^d) = \{g \in L^2(\mathbb{R}^d) : V_\phi g \in L^1(\mathbb{R}^{2d})\}.$$

See [25, Chapters 11–13] for modulation spaces, Wiener amalgams, the Feichtinger algebra, and their role in Gabor analysis. The relevant inclusions are

$$\begin{aligned} C_c^\infty(\mathbb{R}^d) &\subset \mathcal{S}(\mathbb{R}^d) \subset S_0(\mathbb{R}^d) = W(\mathcal{F}L^1, L^1)(\mathbb{R}^d) \\ &\subset W_0(\mathbb{R}^d) \subset W(C, \ell^1)(\mathbb{R}^d) \subset C_Z(\mathbb{R}^d) \subset L^2(\mathbb{R}^d). \end{aligned}$$

The Feichtinger algebra $S_0 = W(\mathcal{F}L^1, L^1)$ is invariant under every metaplectic operator; see Section 13, [25, Section 9.4], and [22].

Part (a) records the sufficiency of (SG) recalled after Theorem 1.3; the following theorem also collects the more explicit existence and sharp nonexistence statements proved below.

Theorem 1.7. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a full-rank phase-space lattice.*

- (a) *If Λ satisfies (SG), then there exists $g \in \mathcal{S}(\mathbb{R}^d)$ such that $\mathcal{G}(g, \Lambda)$ is a Parseval Gabor frame.*
- (b) *If $\Lambda = \Gamma \times \Phi$ is separable and satisfies (SG), then there exists a nonnegative $g \in C_c^\infty(\mathbb{R}^d)$ such that $\mathcal{G}(g, \Gamma \times \Phi)$ is a Parseval frame and*

$$\text{supp } g \cap (\text{supp } g + \xi^*) = \emptyset, \quad \xi^* \in \Phi^* \setminus \{0\}, \quad (1.3)$$

$$\sum_{\gamma \in \Gamma} |g(x + \gamma)|^2 = \text{covol}(\Phi), \quad x \in \mathbb{R}^d. \quad (1.4)$$

Equivalently, $\text{covol}(\Phi)^{-1}|g|^2$ is a smooth Γ -partition of unity. In particular,

$$g(x)g(x - \xi^*) = 0, \quad x \in \mathbb{R}^d, \quad \xi^* \in \Phi^* \setminus \{0\}.$$

There is also an unsmoothed polyhedral alternative: one may choose a nonnegative Parseval window $g_{\text{pa}} \in C_c(\mathbb{R}^d)$ such that $\text{covol}(\Phi)^{-1}g_{\text{pa}}^2$ is a continuous piecewise-affine Γ -partition of unity and (1.3)–(1.4) hold with g_{pa} in place of g .

Alternatively, one may choose $g \in \mathcal{S}(\mathbb{R}^d)$ so that $\widehat{g} \in C_c^\infty(\mathbb{R}^d)$ is nonnegative and

$$\text{supp } \widehat{g} \cap (\text{supp } \widehat{g} + \gamma^*) = \emptyset, \quad \gamma^* \in \Gamma^* \setminus \{0\}, \quad (1.5)$$

$$\sum_{\phi \in \Phi} |\widehat{g}(\omega + \phi)|^2 = \text{covol}(\Gamma), \quad \omega \in \mathbb{R}^d. \quad (1.6)$$

Equivalently, $\text{covol}(\Gamma)^{-1}|\widehat{g}|^2$ is a smooth Φ -partition of unity. There is likewise an unsmoothed Fourier-side alternative: one may choose a Parseval window $g_{\text{fpa}} \in L^2(\mathbb{R}^d)$ such that $\widehat{g}_{\text{fpa}} \in C_c(\mathbb{R}^d)$ and $\text{covol}(\Gamma)^{-1}|\widehat{g}_{\text{fpa}}|^2$ is a continuous piecewise-affine Φ -partition of unity, and (1.5)–(1.6) hold with g_{fpa} in place of g .

(c) Let $a > 0$ and

$$\Lambda = aS\mathbb{Z}^{2d}, \quad S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2d, \mathbb{R}),$$

and assume that Λ satisfies (SG). If $A = 0$ or $B = 0$, a Parseval frame window $g \in C_c^\infty(\mathbb{R}^d)$ can be chosen. Alternatively, one may choose a compactly supported continuous Parseval window for which $|g|^2$ is piecewise affine. If $C = 0$ or $D = 0$, a Parseval frame window can be chosen so that $\widehat{g} \in C_c^\infty(\mathbb{R}^d)$; alternatively, $\widehat{g}$ may be chosen compactly supported and continuous with $|\widehat{g}|^2$ piecewise affine.

(d) Let

$$D = \text{diag}(b_1/a_1, \dots, b_d/a_d), \quad (a_i, b_i) = 1,$$

where the a_i, b_i are positive integers, let $\Lambda_D = D\mathbb{Z}^d \times \mathbb{Z}^d$, and put

$$N = \prod_{i=1}^d a_i, \quad R = \prod_{i=1}^d b_i.$$

If Λ_D does not satisfy (SG), then there is no $g \in C_Z(\mathbb{R}^d)$ for which $\mathcal{G}(g, \Lambda_D)$ is a frame. In particular, no window in the Fourier-invariant Wiener space $W_0(\mathbb{R}^d)$, or even in the larger classical Wiener amalgam $W(C, \ell^1)(\mathbb{R}^d)$, can generate such a frame.

(e) If Λ does not satisfy (SG), then there is no window

$$g \in W(\mathcal{F}L^1, L^1)(\mathbb{R}^d) = S_0(\mathbb{R}^d)$$

for which $\mathcal{G}(g, \Lambda)$ is a frame. This is Theorem 1.4.

Consequently, a full-rank phase-space lattice admits a one-window Gabor frame with a window in the metaplectically invariant Wiener amalgam $W(\mathcal{F}L^1, L^1) = S_0$ if and only if the two inequalities in (SG) hold.

The block conditions in part (c) are sufficient, but they are not necessary for a compactly supported smooth Parseval window. Let

$$a = \frac{71}{100}, \quad \Gamma_0 = a \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \mathbb{Z}^2.$$

Appendix B constructs $g_0 \in C_c^\infty(\mathbb{R})$ such that $\mathcal{G}(g_0, \Gamma_0)$ is an orthonormal system. Its adjoint lattice is

$$\Lambda_0 = \Gamma_0^\circ = \frac{1}{2a} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \mathbb{Z}^2 = cS\mathbb{Z}^2, \quad c = \frac{1}{\sqrt{2}a}, \quad S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

All four scalar blocks of S are nonzero. Nevertheless, $g = \sqrt{\text{covol}(\Lambda_0)} g_0$ generates a Parseval frame on Λ_0 . The relevant arithmetic values are

$$\text{covol}(\Lambda_0) = \frac{5000}{5041}, \quad \nu(\Lambda_0) = 5041, \quad \nu(\Lambda_0^\circ) = 5000.$$

The full elementary construction and verification are given in Appendix B.

The detailed proofs of parts (b)–(e) are given in Part B. We next state the corresponding multiple-window result.

For $\mathbf{g} = (g_1, \dots, g_q)$, write

$$\mathcal{G}(\mathbf{g}, \Lambda) = \{\pi(\lambda)g_j : \lambda \in \Lambda, 1 \leq j \leq q\}.$$

Theorem 1.8. *Let $q \geq 1$ and let $\Lambda \subset \mathbb{R}^{2d}$ be a full-rank phase-space lattice. There exist $g_1, \dots, g_q \in \mathcal{S}(\mathbb{R}^d)$ such that $\mathcal{G}(\mathbf{g}, \Lambda)$ is a Parseval frame if and only if*

$$\text{covol}(\Lambda) < q \quad \text{and, if } \Lambda \text{ is symplectically rational,} \quad q v(\Lambda) \geq v(\Lambda^\circ) + d. \quad (\text{mSG}_q)$$

If Λ is symplectically rational, the minimum number of Schwartz windows is

$$q_{\min}^{\mathcal{S}}(\Lambda) = \left\lceil \frac{v(\Lambda^\circ) + d}{v(\Lambda)} \right\rceil.$$

If Λ is not symplectically rational, it is

$$q_{\min}^{\mathcal{S}}(\Lambda) = \lfloor \text{covol}(\Lambda) \rfloor + 1.$$

On a diagonal representative of a symplectically rational lattice, the necessity of (mSG_q) remains valid under the weaker assumptions $g_1, \dots, g_q \in C_Z(\mathbb{R}^d)$. On an arbitrary symplectically rational lattice it remains valid for $g_1, \dots, g_q \in S_0(\mathbb{R}^d)$, by metaplectic invariance. In fact, the if-and-only-if statement above remains true with $\mathcal{S}(\mathbb{R}^d)$ replaced by $S_0(\mathbb{R}^d)$.

Outline of the paper

Section 2 summarizes the relevant literature. Part A proves Theorem 1.1: it develops the configuration-space construction in the commensurable, fully dense, and partially dense cases, and then proves necessity. Part B proves the one-window and multiwindow Gabor results, beginning with Zak-transform and lattice preliminaries and ending with the rational obstruction and metaplectic transfer. Appendix A gives a first-principles common-zero proof, and Appendix B gives a compactly supported Parseval example outside the block-zero cases.

2. EARLIER RESULTS AND RELATION TO THE LITERATURE

This section records the strands of the literature that motivate the geometric and Gabor-theoretic arguments used later, without interrupting the proofs in Parts A and B.

2.1. Tiling and packing by lattices

The relation between translational tilings, packings, and orthogonal exponentials has a long history. Kolountzakis developed the viewpoint that orthogonality and completeness can be encoded as packing and tiling identities for suitable nonnegative functions [36]. Fourier methods for lattice tilings by cubes and related polyhedral sets appear in [35]. These works are part of the broader connection between translational tiling and spectral questions.

Han and Wang [30] proved that any two full-rank lattices of equal covolume have a common measurable fundamental domain and, more generally, that if $\text{covol}(L) \leq \text{covol}(M)$ then there is a measurable set which tiles by L and packs by M . Essential predecessors and extensions for simultaneous tiling include the multi-lattice problem [34], compactly supported functions tiling with several lattices [37], and common domains for more general group actions [14].

Grepstad and Kolountzakis [23] proved that two Euclidean lattices of equal covolume have a bounded common fundamental domain, which can be chosen as a finite union of polytopes. A direct bounded measurable construction was subsequently given by Spyridakis [43]. Grepstad, Kolountzakis, and Spyridakis [24] established the corresponding bounded tile/packing statement under the strict covolume inequality. The present paper

asks for the stronger metric conclusion that a positive neighborhood of the tile packs. This is not a null-set refinement: it requires a uniform gap between every pair of nontrivial packing translates.

Theorem 1.1 gives the sharp answer. In the commensurable case a dimension-dependent finite-index gap appears; when the lattice sum is not discrete, the criterion is the strict covolume inequality. The proofs in these two regimes are different and are developed separately in Part A.

2.2. Gabor frames, regular windows, and Balian–Low phenomena

A standard general reference for Gabor frames, duality, density, the Zak transform, modulation spaces, and the Balian–Low theorem is Gröchenig’s monograph [25]. The geometric use of fundamental domains goes back to Han and Wang [30]; related constructive results include the painless expansions of Daubechies, Grossmann, and Meyer [12] and the smooth common-domain construction of Pfander, Rashkov, and Wang [41]. The duality and finite Zak-matrix tools used later originate in work of Janssen, Ron–Shen, and Zibulski–Zeevi [33, 42, 45].

For classical and modern forms of the Balian–Low phenomenon, see [3, 4, 6–10, 26, 38]. Gabor expansions on modulation and amalgam spaces, and stability under changes of lattice, are treated in [17, 18, 27, 28]. For related work on localized or continuous frames, frame operators, and regular or compactly supported dual Gabor windows, see [2, 11, 20, 44]. Structured Gabor windows and sharp frame-set questions for Hermite or totally positive windows are developed, among other places, in [1, 29]. For general phase-space lattices, operator-algebraic methods interpret Gabor frames through Heisenberg modules and noncommutative tori [32, 39]. The Schwartz existence result of Enstad–Thiel–Vilalta [16], already summarized in Section 1, is the main external positive result used in Part B. Symplectic normal forms and metaplectic covariance provide the reduction from rational lattices to diagonal models; see [19, 22, 40]. The present paper combines these ideas with a rectangular common-zero argument to obtain the sharp arithmetic gap.

Part A

Tilings and ε -packings3. PLANAR EXAMPLES FOR THE TILING- ε -PACKING THEOREM

This section gives a graphical progression from common fundamental domains, through ordinary tiling–packing pairs, to the buffered packing problem of Theorem 1.1. In every tiling or packing picture, the unshifted fundamental domain is the unique dark blue copy. Other translates are assigned successive colors from the twenty-color palette fixed in the preamble.

3.1. The simultaneous tiling problem

The classical simultaneous tiling problem asks whether two lattices of the same covolume admit a single measurable fundamental domain. It appears in [14, 23, 30, 41]. Figure 1 begins with separate fundamental domains for

$$L_1 = 2\mathbb{Z} \times 2\mathbb{Z}, \quad L_2 = 4\mathbb{Z} \times \mathbb{Z},$$

and then shows that the parallelogram spanned by $(0, 1)$ and $(4, 2)$ is a fundamental domain for both lattices.

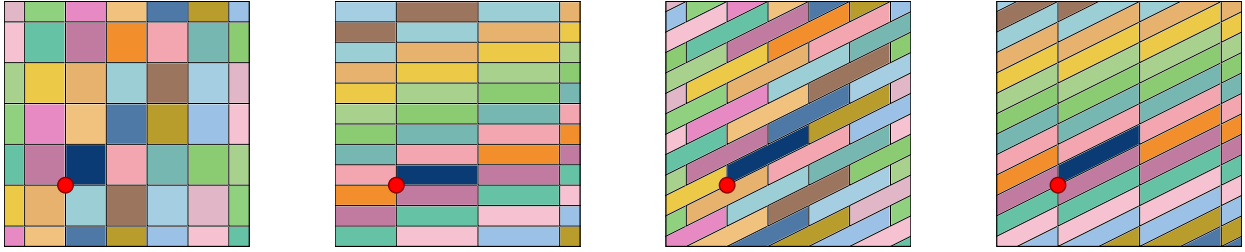


FIGURE 1. The simultaneous tiling problem for two lattices of covolume 4. From left to right: the square $[0, 2]^2$ tiled by $2\mathbb{Z} \times 2\mathbb{Z}$; the rectangle $[0, 4] \times [0, 1]$ tiled by $4\mathbb{Z} \times \mathbb{Z}$; and the common parallelogram spanned by $(0, 1)$ and $(4, 2)$ tiled by the two respective lattices. Here and in the following, the origin of $\mathbb{R}^2$ is marked with a red point.

We next describe a finite cube procedure that captures the common-domain problem for integer lattices. Let $T, P \subset \mathbb{Z}^d$ satisfy

$$T, P \subset \mathbb{Z}^d, \quad T + P = \mathbb{Z}^d. \quad (3.1)$$

Put

$$m = [\mathbb{Z}^d : T], \quad n = [\mathbb{Z}^d : P],$$

and enumerate the quotient groups by

$$\mathbb{Z}^d / T = \{R_0, R_1, \dots, R_{m-1}\}, \quad R_0 = T, \quad (3.2)$$

$$\mathbb{Z}^d / P = \{Q_0, Q_1, \dots, Q_{n-1}\}, \quad Q_0 = P. \quad (3.3)$$

Assume first that T and P have the same covolume. For sublattices of $\mathbb{Z}^d$, this is equivalent to $m = n$, because $\text{covol}(T) = [\mathbb{Z}^d : T] \text{covol}(\mathbb{Z}^d) = m$ and similarly $\text{covol}(P) = n$. Let $F = [0, 1]^d$. To obtain a common fundamental domain built from unit cubes, choose

addresses $z_0, \dots, z_{m-1} \in \mathbb{Z}^d$ so that the classes $z_i + T$ run through $\mathbb{Z}^d/T$ exactly once and the classes $z_i + P$ run through $\mathbb{Z}^d/P$ exactly once. Then

$$\Omega(Z) = \bigcup_{i=0}^{m-1} (F + z_i) \quad (3.4)$$

is a fundamental domain for both lattices. Indeed, the half-open cubes $F + z$, $z \in \mathbb{Z}^d$, partition $\mathbb{R}^d$, and the selected addresses give exactly one representative in every coset modulo either lattice.

Only the two residue classes of an address matter for this argument. If $h \in T \cap P$, then replacing z_i by $z_i + h$ changes neither class and therefore preserves both tiling properties. This freedom lets us move the individual cubes without changing the underlying finite data.

A configuration table records these data. The sets R_i index its rows, the sets Q_j index its columns, and the cell in row R_i and column Q_j is the address class

$$\mathcal{A}_{i,j} = R_i \cap Q_j = \{z \in \mathbb{Z}^d : z + T = R_i, z + P = Q_j\}. \quad (3.5)$$

Every address class is nonempty. Indeed, if $R_i = a + T$ and $Q_j = b + P$, write $b - a = t + p$ with $t \in T$ and $p \in P$; then $a + t = b - p$ belongs to $R_i \cap Q_j$. Moreover, for any $z_{i,j} \in \mathcal{A}_{i,j}$,

$$\mathcal{A}_{i,j} = z_{i,j} + (T \cap P). \quad (3.6)$$

Thus every cell contains infinitely many actual addresses. In the figures, a cell may be divided into four subcells to display four chosen representatives; subcell positions are read in the order lower left, lower right, upper left, upper right. The representatives used for each guiding lattice pair are specified when that pair is introduced.

Definition 3.1. A *hard configuration* is a finitely supported function $\lambda : \mathbb{Z}^d \rightarrow \{0, 1\}$ such that every row contains exactly one occupied address,

$$\sum_{z \in R_i} \lambda(z) = 1, \quad 0 \leq i \leq m-1, \quad (3.7)$$

and every column contains at most one occupied address,

$$\sum_{z \in Q_j} \lambda(z) \leq 1, \quad 0 \leq j \leq n-1. \quad (3.8)$$

Equivalently, the occupied addresses meet every row once and every column at most once. When $m = n$, every column is occupied exactly once.

For a concrete example, take

$$T = 4\mathbb{Z} \times \mathbb{Z}, \quad P = \mathbb{Z} \times 4\mathbb{Z}.$$

The table in Figure 2 uses the non-diagonal matching

$$R_0 \leftrightarrow Q_1, \quad R_1 \leftrightarrow Q_3, \quad R_2 \leftrightarrow Q_0, \quad R_3 \leftrightarrow Q_2.$$

In the subcell order fixed above, the four displayed representatives are $(0, 1)$, $(5, 3)$, $(2, 4)$, and $(7, 6)$. They occur, respectively, in the lower-left, lower-right, upper-left, and upper-right positions of their occupied main cells. Subtracting suitable vectors of $T \cap P = 4\mathbb{Z} \times 4\mathbb{Z}$ gives the compact representatives

$$Z_{4,4} = \{(0, 1), (1, 3), (2, 0), (3, 2)\}. \quad (3.9)$$

Thus

$$\Omega_{4,4} = \bigcup_{z \in Z_{4,4}} ([0, 1)^2 + z)$$

is a common fundamental domain.

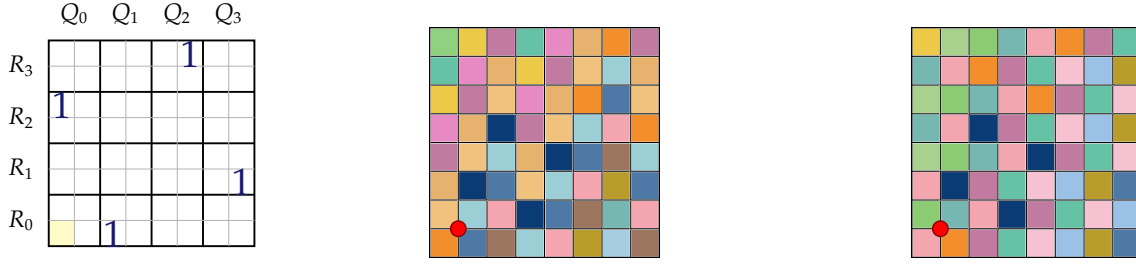


FIGURE 2. A cube-built common fundamental domain. From left to right: a non-diagonal 4×4 hard configuration, with four sample addresses displayed in four different subcells; the tiling by T ; and the tiling by P , both obtained from the compact representatives in (3.9).

3.2. The tiling and packing problem

Suppose now that $T, P \subset \mathbb{Z}^d$, $T + P = \mathbb{Z}^d$, and $\text{covol}(T) < \text{covol}(P)$, equivalently $m < n$. A hard configuration uses every row and only m of the n columns. The cube union (3.4) is a T -fundamental domain and packs with P .

For a planar example take $T = 4\mathbb{Z} \times \mathbb{Z}$ and $P = \begin{pmatrix} 1 & 0 \\ 1 & 5 \end{pmatrix} \mathbb{Z}^2$. Then $T + P = \mathbb{Z}^2$, $m = 4$, and $n = 5$. The rows are the first-coordinate classes modulo 4, while the columns are given by $y - x$ modulo 5. Choose the addresses

$$Z_{4,5} = \{(0,0), (1,0), (2,0), (3,0)\}, \quad \Omega_{4,5} = [0,4) \times [0,1). \quad (3.10)$$

They occupy $R_0 \cap Q_0$, $R_1 \cap Q_4$, $R_2 \cap Q_3$, and $R_3 \cap Q_2$. All four entries are displayed in the lower-left address subcell, and one column remains unoccupied. The set $\Omega_{4,5}$ tiles by T and packs by P ; the shear in P makes the vacant regions stagger rather than line up in one horizontal strip.

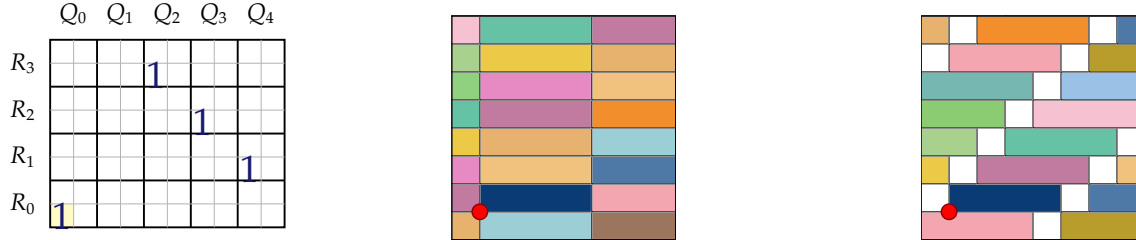


FIGURE 3. The tiling and packing problem. From left to right: a 4×5 hard configuration whose entries all use the lower-left address subcell and leave one column unoccupied; the tiling by $T = 4\mathbb{Z} \times \mathbb{Z}$; and the staggered packing by $P = \begin{pmatrix} 1 & 0 \\ 1 & 5 \end{pmatrix} \mathbb{Z}^2$.

3.3. The tiling and ε -packing problem

The stronger goal is to find a T -fundamental domain Ω and $\varepsilon > 0$ such that $\Omega + B_\varepsilon$ packs with P , so that nontrivial packing translates are separated by a positive gap.

A second solution to the same lattice pair is obtained from the soft field. Start from the unit square $F = [0,1)^2$, partition it into the finitely many half-open polygonal pieces selected by the first-hit construction in Subsection 6.1, and translate those pieces by integer vectors. Figure 5 displays this cube-derived domain for the alternative first-hit enumeration $((1,0), (0,-1), (-1,-1), (0,0), (-1,0))$.

The analogous buffered problem cannot be solved for

$$T = \mathbb{Z}^2, \quad P = \mathbb{Z} \times 2\mathbb{Z}.$$

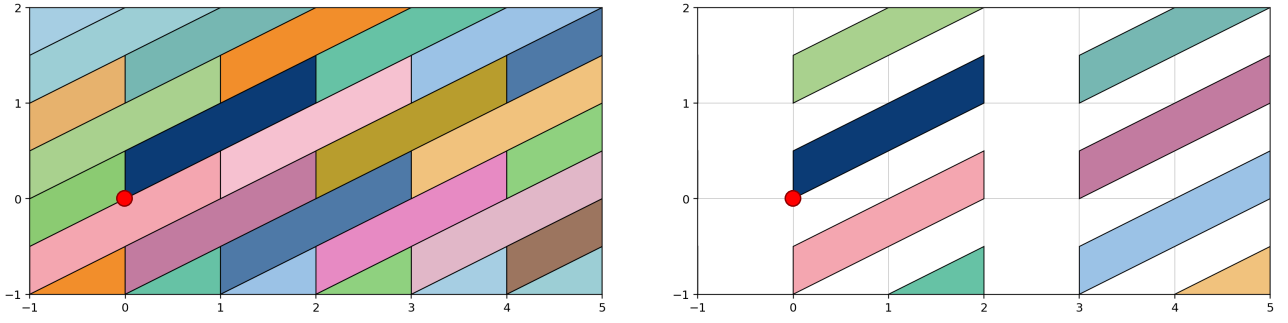


FIGURE 4. A direct buffered construction for $T = \mathbb{Z}^2$ and $P = 3\mathbb{Z} \times \mathbb{Z}$. The parallelogram is spanned by $(0, 1/2)$ and $(2, 1)$. Left: the T -tiling. Right: the buffered P -packing.

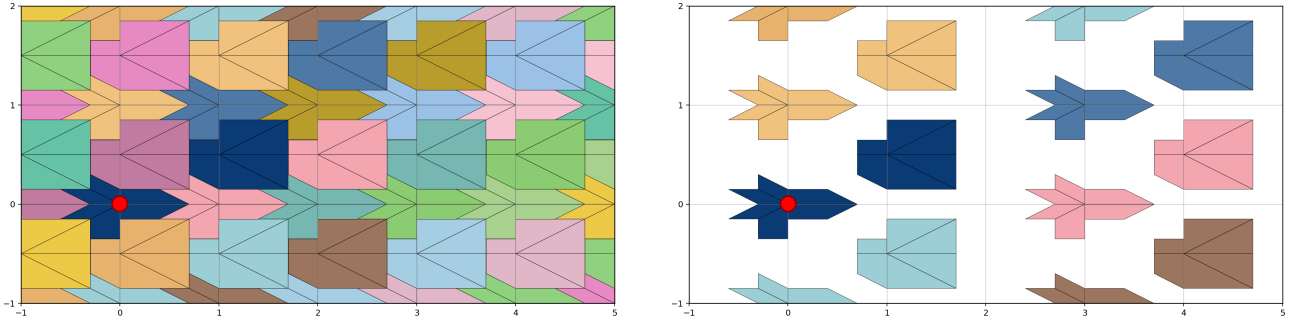


FIGURE 5. The cube-derived solution of the $\mathbb{Z}^2$ - $3\mathbb{Z} \times \mathbb{Z}$ problem obtained from the alternative first-hit enumeration above. From left to right: the exact tiling by $\mathbb{Z}^2$ and the buffered packing by $3\mathbb{Z} \times \mathbb{Z}$. The detailed construction in Subsection 6.1 retains the standard enumeration.

This obstruction appears in the geometric Gabor-frame work of Pfander, Rashkov, and Wang [41], and it also follows from the necessity theorem proved later in this paper. The best corresponding parallelogram picture gives only ordinary packing, with boundary contact.

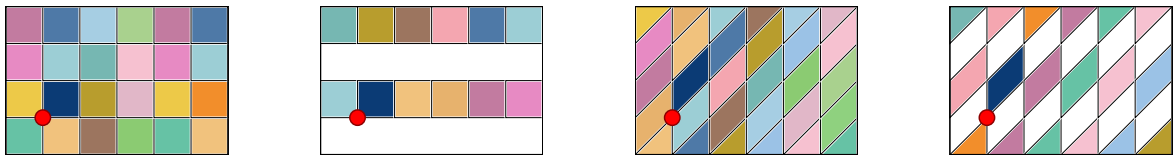


FIGURE 6. Two ordinary tiling–packing solutions for $T = \mathbb{Z}^2$ and $P = \mathbb{Z} \times 2\mathbb{Z}$. From left to right: the unit-square tiling and packing; and the tiling and packing by the parallelogram spanned by $(0, 1)$ and $(1, 1)$. Both packings have boundary contact, so neither has a positive buffer.

For other pairs at the sharp planar index gap, a geometric construction may succeed without being described by one fixed hard table. In the next example the fundamental domain is cut into polyhedral pieces and different pieces are translated by different lattice vectors. Equivalently, the associated hard choice varies piecewise with the fractional position.

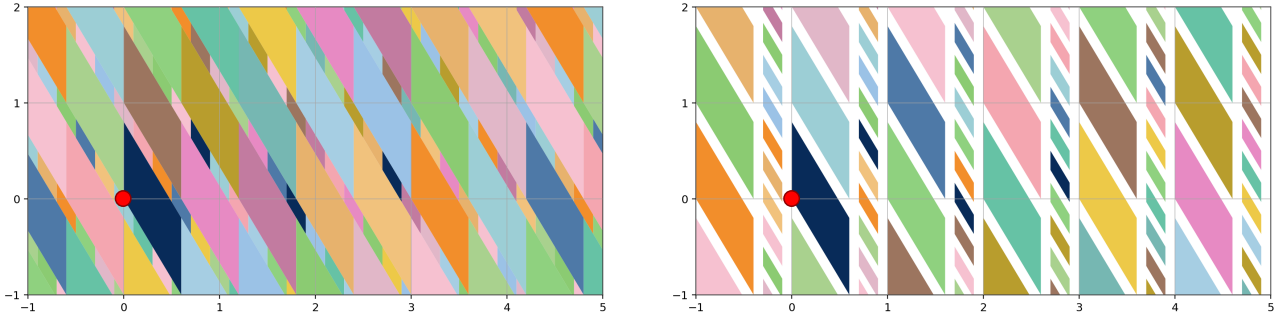


FIGURE 7. The sharp planar integer example $T = 3\mathbb{Z} \times \mathbb{Z}$ and $P = 5\mathbb{Z} \times \mathbb{Z}$, for which $m = 3$ and $n = 5 = m + 2$. The displayed cut-and-translate construction is not based on one fixed hard configuration. Left: the T -tiling. Right: the buffered P -packing.

The main theorem classifies all lattice pairs for which the buffered problem can be solved. In the normalized commensurable case the criterion is

$$n \geq m + d,$$

which becomes $n \geq m + 2$ in the plane. The remaining sections prove sufficiency by replacing a fixed hard table with a continuous field of soft configurations, then applying a first-hit selector to recover the actual polyhedral fundamental domain; see Sections 4 and 5.

3.4. The noncommensurable case

The noncommensurable case has a different source of flexibility. The connected component of $\overline{T + P}$ can be subdivided into arbitrarily many small source packets and target slots. These labels refine the finite data from the discrete quotient, so the configuration table can be enlarged until the required surplus is available. This is made precise in Sections 8 and 9.

As a concrete irrational example, let

$$T = \mathbb{Z}^2, \quad P = \sqrt{2}\mathbb{Z} \times \mathbb{Z}.$$

Then $\overline{T + P} = \mathbb{R} \times \mathbb{Z}$ and $\text{covol}(T) = 1 < \sqrt{2} = \text{covol}(P)$. Start from the T -fundamental parallelogram spanned by $(0, 1)$ and $(1, -1)$. Keep the smaller main parallelogram spanned by $(0, 9/10)$ and $(1, -1)$, divide the remaining thin strip into five vertical slices, and translate these slices by $(4, 0), (1, 0), (5, 0), (2, 0), (6, 0) \in T$. The tiling property is unchanged, and periodization by P places the slices in separated gaps.

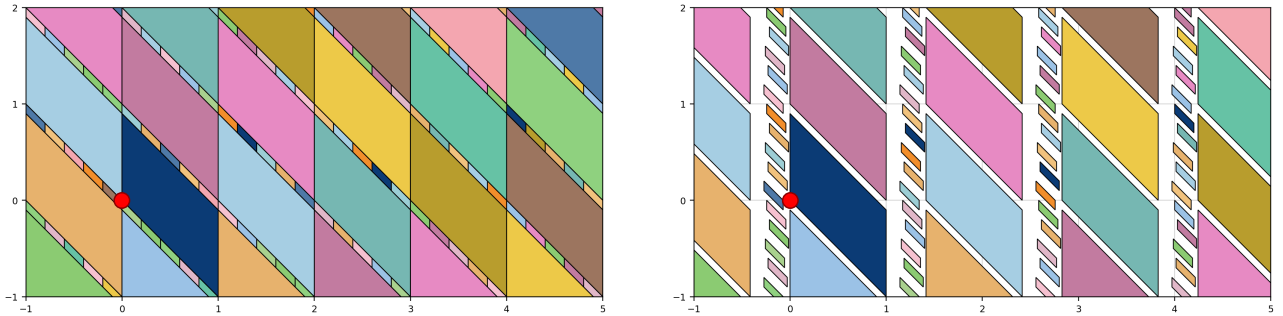


FIGURE 8. A direct cut-and-translate construction for $T = \mathbb{Z}^2$ and $P = \sqrt{2}\mathbb{Z} \times \mathbb{Z}$. Left: the T -tiling. Right: the buffered P -packing. The unshifted fundamental domain is dark blue.

In the commensurable case, the finite table is fixed by the quotient groups. In the noncommensurable case, the connected factor supplies as many packet and slot labels as needed; the resulting enlarged table is then filled by the same equivariant configuration-space method used later.

4. SOFT CONFIGURATIONS

We now overcome the limitations of hard configurations by allowing the selected addresses to vary continuously and by replacing a single hard table with a field of weighted tables.

4.1. Partitions of unity generating tiling and ε -packing sets Ω

Throughout this subsection, φ is assumed nonnegative, continuous, and compactly supported. Continuity makes its positive superlevel sets closed, while compact support makes them compact and ensures that only finitely many translates meet a fixed fundamental parallelepiped. We call φ a T -partition of unity when its T -translates are locally finite and

$$\sum_{t \in T} \varphi(x + t) = 1 \quad (x \in \mathbb{R}^d). \quad (4.1)$$

The goal of Section 5 is to construct such a function that is compactly supported and continuous piecewise affine, and also satisfies

$$\varphi(x)\varphi(x - p) = 0, \quad p \in P \setminus \{0\}. \quad (4.2)$$

The partition-of-unity identity implies $\{\varphi > 0\} + T = \mathbb{R}^d$. The second identity gives pointwise disjointness of nontrivial P -translates. To obtain a positive packing buffer we pass to a compact positive superlevel set.

Write $T = A\mathbb{Z}^d$ and let

$$F = A[0, 1)^d$$

be the standard half-open T -fundamental parallelepiped. Since $\text{supp } \varphi$ is compact and the sets $F + t$, $t \in T$, partition $\mathbb{R}^d$, there are $t_1, \dots, t_N \in T$ such that

$$\text{supp } \varphi \subset \bigcup_{j=1}^N (F + t_j).$$

Consequently, for every $x \in F$,

$$1 = \sum_{t \in T} \varphi(x + t) = \sum_{j=1}^N \varphi(x + t_j). \quad (4.3)$$

Therefore, at least one term contributes at least $1/N$ to the sum. Put

$$K = \{y \in \mathbb{R}^d : \varphi(y) \geq 1/N\}, \quad D_j = \{x \in F : x + t_j \in K\}.$$

The sets $D_1, \dots, D_N$ cover F . Since φ is continuous and piecewise affine on a finite polyhedral subdivision of its compact support, K is a compact finite union of polyhedra.

Choose the first admissible translate in the fixed order by setting

$$E_1 = D_1, \quad E_j = D_j \setminus \bigcup_{k < j} D_k \quad (2 \leq j \leq N), \quad \Omega = \bigcup_{j=1}^N (E_j + t_j). \quad (4.4)$$

The sets E_j form a disjoint partition of F . Hence every T -coset has exactly one representative in Ω , so $\Omega + T$ is an exact tiling. After refining the finitely many polyhedral subdivisions on which the functions $x \mapsto \varphi(x + t_j)$ are affine, each E_j is a finite union of half-open polyhedral pieces. Thus Ω is a finite union of bounded half-open polyhedra.

Equations (4.1) and (4.2) imply

$$K \cap (K + p) = \emptyset, \quad p \in P \setminus \{0\}.$$

Because K is compact and P is discrete,

$$\delta := \inf_{p \in P \setminus \{0\}} \text{dist}(K, K + p) > 0.$$

Since $\Omega \subset K$, for every $0 < \varepsilon < \delta/2$, the set $\Omega + B_\varepsilon$ packs with P . Section 5 uses the sharper row–column threshold $1/(n - m + 1)$ instead of the coarser $1/N$ used here.

We retain the normalization and notation of Section 3. For all configuration pictures below, the representatives in the four subcell positions are fixed by the guiding pair $T = \mathbb{Z} \times 4\mathbb{Z}$, $P = 6\mathbb{Z} \times \mathbb{Z}$; thus $R_i = (0, i) + T$, $Q_j = (j, 0) + P$, $\mathcal{A}_{i,j} = (j, i) + (6\mathbb{Z} \times 4\mathbb{Z})$, and the four subcells of $\mathcal{A}_{i,j}$ represent (j, i) , $(j + 6, i)$, $(j, i + 4)$, and $(j + 6, i + 4)$.

4.2. Why hard configurations are insufficient

The preceding examples show that fixed hard tables can work for some pairs. Nevertheless, an arbitrarily large surplus $n - m$ does not force the unit-square construction to have a positive buffer for a given pair. The following elementary family makes this limitation explicit.

Remark 4.1. An arbitrarily large column surplus does not guarantee a positive buffer for the unit-square hard construction. For any $1 \leq m < n$, take $T = m\mathbb{Z} \times \mathbb{Z}$ and $P = \mathbb{Z} \times n\mathbb{Z}$. A hard configuration gives a union of unit cubes that tiles by T and packs by P . Since $(1, 0) \in P$, the closure of every selected cube meets the closure of its translate by $(1, 0)$. The distance to a nonzero P -translate is therefore zero, independently of $n - m$.

4.3. Smith normal form reduction

The ordinary Smith normal form recalled in Section 13 implies that, after a further common invertible linear change of variables, the commensurable pair may be put in the diagonal form

$$T = \text{diag}(b_1, \dots, b_d)\mathbb{Z}^d, \quad P = \text{diag}(a_1, \dots, a_d)\mathbb{Z}^d, \quad \gcd(a_\ell, b_\ell) = 1.$$

In these coordinates, $\mathbb{Z}^d/T \cong \prod_{\ell=1}^d \mathbb{Z}/b_\ell\mathbb{Z}$, $m = \prod_{\ell=1}^d b_\ell$, $\mathbb{Z}^d/P \cong \prod_{\ell=1}^d \mathbb{Z}/a_\ell\mathbb{Z}$, and $n = \prod_{\ell=1}^d a_\ell$. If R_i corresponds to (α_ℓ) and Q_j corresponds to (q_ℓ) , the coordinatewise Chinese remainder theorem gives a unique representative $c(i, j)$ modulo $\text{diag}(a_1b_1, \dots, a_db_d)\mathbb{Z}^d$, and

$$\mathcal{A}_{i,j} = c(i, j) + \text{diag}(a_1b_1, \dots, a_db_d)\mathbb{Z}^d.$$

The reader may keep this diagonal model in mind. We do not impose it in the proof, because the row–column construction works directly under (3.1).

4.4. Soft configurations

The hard configurations of Definition 3.1 are the vertices of the continuous configuration space introduced below. They choose one actual address in every row, with the chosen addresses in pairwise distinct columns. Since every address class is nonempty, a hard configuration exists whenever $n \geq m$.

With the guiding address convention above, the supports of the two hard configurations are, respectively,

$$\{(0, 0), (2, 1), (4, 2), (5, 3)\} \quad \text{and} \quad \{(1, 0), (3, 1), (0, 2), (4, 3)\}. \quad (4.5)$$

Each support meets every R_i once and every Q_j at most once.

At the level of row–column choices, the finite cut-and-translate viewpoint is closely related to the lattice packing and tiling arguments in [35, 36]. If a row occupies one column and another column is vacant, the row can be moved to an address in the vacant

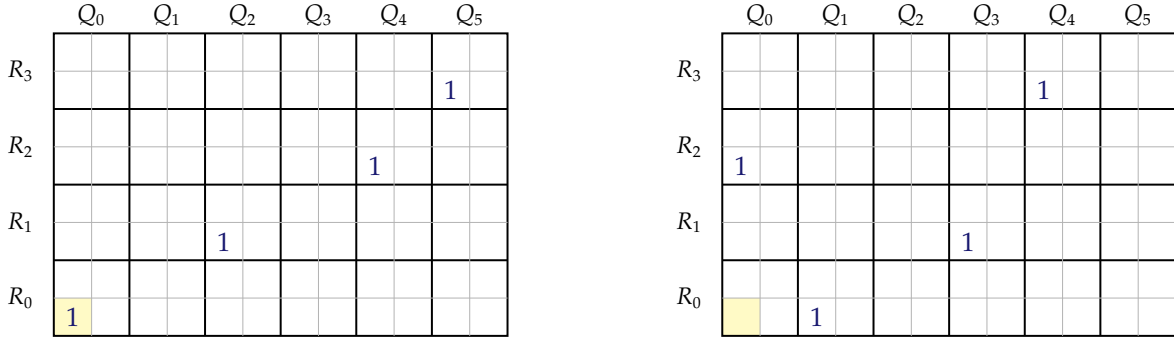


FIGURE 9. Two hard configurations with four rows and six columns. Every main cell is an address class $\mathcal{A}_{i,j} = R_i \cap Q_j$, and its four subcells represent four possible addresses. In each panel, exactly one subcell in every row has value 1, and the four occupied main cells lie in distinct columns. The two unused columns provide the vacancies for elementary hard moves.

column. Its previous column then becomes vacant. Consequently, if $n \geq m + 1$, any two hard configurations can be joined by a sequence of such elementary moves through a vacant column.

Hard configurations are discrete. To let configurations vary continuously, we allow each row to distribute its unit mass among finitely many addresses.

Definition 4.2. A *soft configuration* is a finitely supported function

$$\lambda : \mathbb{Z}^d \longrightarrow [0, 1]$$

such that every row has total mass one,

$$\sum_{z \in R_i} \lambda(z) = 1, \quad i = 0, \dots, m-1, \quad (4.6)$$

and distinct positive addresses never lie in the same column:

$$z \neq z', \quad z + P = z' + P \implies \lambda(z)\lambda(z') = 0. \quad (4.7)$$

We denote the resulting space by $C(T, P)$.

A hard configuration is precisely a soft configuration whose values lie in $\{0, 1\}$. The row condition and the column-exclusion condition are now expressed directly in the ambient group $\mathbb{Z}^d$.

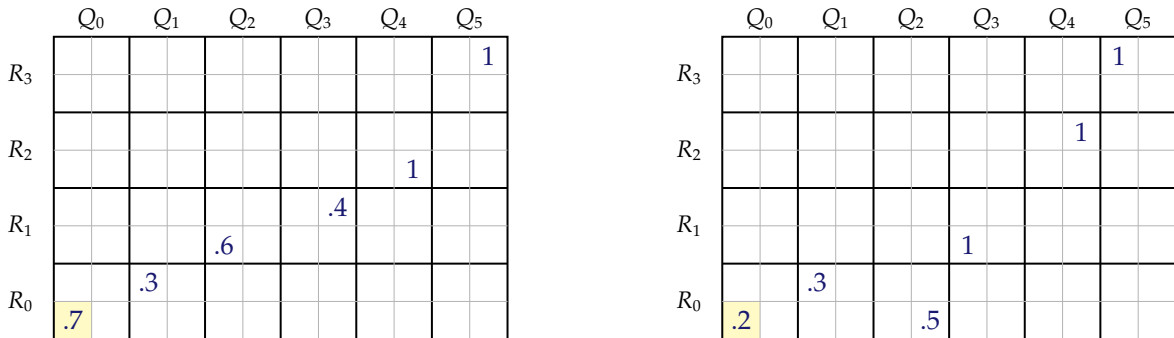


FIGURE 10. Two soft configurations. Empty subcells have weight zero. In every row the displayed values add to 1, while no main column contains positive values belonging to two different rows. In this schematic each main cell is an address class $\mathcal{A}_{i,j}$ and displays four possible addresses. The column-exclusion rule allows at most one positive subcell anywhere in each column.

The same guiding pair also provides the first concrete illustration of why weights are useful. The following legal weighted table occurs in the later equivariant field and produces the buffered construction that a fixed hard table cannot supply.

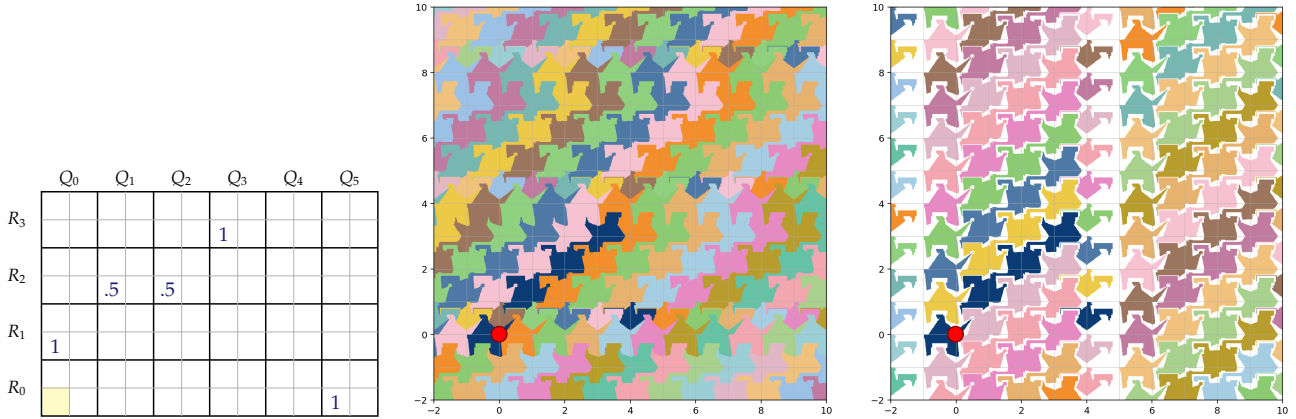


FIGURE 11. A guiding soft-configuration example for $T = \mathbb{Z} \times 4\mathbb{Z}$ and $P = 6\mathbb{Z} \times \mathbb{Z}$. From left to right: one legal weighted configuration occurring in the configuration field; the resulting exact T -tiling; and the buffered P -packing. The complete construction is given in the second example in Subsection 6.2.

An admissible support pattern is a tuple

$$S = (S_0, \dots, S_{m-1}),$$

where every $S_i \subset R_i$ is finite and nonempty and no two addresses in $\bigcup_i S_i$ lie in the same column Q_j . The configurations supported in S form the convex cell

$$C_S = \prod_{i=0}^{m-1} \Delta(S_i), \quad (4.8)$$

where $\Delta(S_i)$ is the probability simplex on S_i .

The first configuration in Figure 10 has support pattern $S^A = (S_0^A, S_1^A, S_2^A, S_3^A)$ with

$$\begin{aligned} S_0^A &= \{(0, 0), (1, 4)\}, & S_1^A &= \{(2, 1), (9, 5)\}, \\ S_2^A &= \{(10, 2)\}, & S_3^A &= \{(11, 7)\}. \end{aligned} \quad (4.9)$$

Thus

$$C_{S^A} = \Delta\{(0, 0), (1, 4)\} \times \Delta\{(2, 1), (9, 5)\} \times \Delta^0 \times \Delta^0 \cong \Delta^1 \times \Delta^1 \times \Delta^0 \times \Delta^0.$$

Here $\Delta^0 = \{1\}$ is the probability simplex on a singleton.

The second configuration has support pattern $S^B = (S_0^B, S_1^B, S_2^B, S_3^B)$ with

$$\begin{aligned} S_0^B &= \{(0, 0), (1, 4), (8, 0)\}, & S_1^B &= \{(3, 1)\}, \\ S_2^B &= \{(10, 6)\}, & S_3^B &= \{(5, 7)\}. \end{aligned} \quad (4.10)$$

Hence

$$C_{S^B} = \Delta\{(0, 0), (1, 4), (8, 0)\} \times \Delta^0 \times \Delta^0 \times \Delta^0 \cong \Delta^2 \times \Delta^0 \times \Delta^0 \times \Delta^0.$$

The point, segment, triangle, and square cell types that occur in this four-row, six-column setting are displayed in Figure 12.

We regard every cell as a subset of the vector space $\mathbb{R}^{(\mathbb{Z}^d)}$ of finitely supported real functions on $\mathbb{Z}^d$, by extending its coordinates by zero outside $\bigcup_i S_i$. With this convention,

$$C(T, P) = \bigcup_{S \text{ admissible}} C_S. \quad (4.11)$$

For $\lambda \in C(T, P)$, the support in row R_i is

$$S_i(\lambda) = \{z \in R_i : \lambda(z) > 0\}.$$

The tuple $S(\lambda)$ is admissible, and $C_{S(\lambda)}$ is the unique smallest cell containing λ . Conversely, every point of every C_S satisfies the row-normalization and column-exclusion conditions. Faces arise by setting weights equal to zero.

If S uses s addresses in total, then

$$\dim C_S = \sum_{i=0}^{m-1} (|S_i| - 1) = s - m.$$

Since an admissible pattern uses at most one address in each of the n columns, $s \leq n$, and hence

$$\dim C_S \leq n - m. \quad (4.12)$$

The full space is generally an infinite polyhedral complex, but every map from a finite triangulation that is defined by finitely many vertex values uses only finitely many addresses and therefore lies in a finite subcomplex.

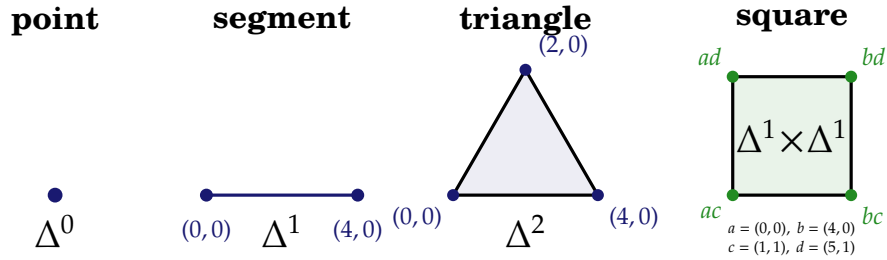


FIGURE 12. The four elementary cell types in the concrete four-row, six-column model. Their lower edges are vertically aligned; the triangle is equilateral and the product cell is drawn as a square. The fixed Δ^0 factors are suppressed. The displayed labels are actual addresses in the guiding classes $\mathcal{A}_{i,j}$, chosen so that the active addresses lie in pairwise distinct columns. In the square, each vertex records the pair selected in rows R_0 and R_1 .

5. COMMENSURABLE SUFFICIENCY IN THE TILING- ε -PACKING THEOREM

We retain without reintroduction the normalization, quotient cosets, address classes, and hard configurations of Section 3, together with the soft configurations of Section 4. Assume throughout this section that

$$n \geq m + d. \quad (5.1)$$

5.1. The fundamental topological result

The commensurable construction repeatedly fills the boundary of a parameter cell. For $k \geq 1$, write $B^k = \{x \in \mathbb{R}^k : \|x\|_2 \leq 1\}$ and $S^{k-1} = \partial B^k$.

Definition 5.1 (Simplexwise affine maps). Let K be a finite simplicial complex, with geometric realization $|K|$. A map $f : |K| \rightarrow C(T, P)$ is *simplexwise affine with respect to K* if, for every simplex $\sigma = [v_0, \dots, v_p]$, the vertex values lie in one common cell C_{S_σ} and

$$f\left(\sum_{i=0}^p t_i v_i\right) = \sum_{i=0}^p t_i f(v_i) \quad \text{whenever } t_i \geq 0 \text{ and } \sum_{i=0}^p t_i = 1.$$

It is *simplexwise affine* if this holds after a finite subdivision of K .

The common-cell condition keeps the affine interpolation inside the soft-configuration space. The formulas agree on common faces, so the map is continuous.

Theorem 5.2. *Assume $n \geq m + d$. For every $1 \leq k \leq d$, every simplexwise affine map from a finitely triangulated S^{k-1} into $C(T, P)$ extends simplexwise affinely over a finitely triangulated B^k without changing the boundary triangulation or its vertex values. No finiteness assumption is imposed on the address classes $\mathcal{A}_{i,j}$; as in Definition 4.2, each individual configuration has finite support. More precisely, for every boundary map there exist finite nonempty sets*

$$B_{i,j} \subset \mathcal{A}_{i,j}, \quad 0 \leq i \leq m-1, \quad 0 \leq j \leq n-1,$$

containing every address that occurs in a boundary vertex value, such that the extension may be chosen to use only addresses in $\bigcup_{i,j} B_{i,j}$.

The proof is deferred to Section 7. There the lattice origin of the address classes is forgotten and a stronger theorem is proved for arbitrary nonempty row–column sets, finite or infinite. The proof reduces every simplexwise affine boundary problem to a finite complete address subsystem before applying the finite polyhedral argument.

5.2. The translation action

For $a \in \mathbb{Z}^d$ and $\lambda \in C(T, P)$, define

$$(\rho_a \lambda)(z) = \lambda(z - a), \quad z \in \mathbb{Z}^d. \quad (5.2)$$

Translation by a permutes the rows R_i , the columns Q_j , and their address classes, so $\rho_a \lambda$ is again a soft configuration. Moreover, $\rho_a \rho_b = \rho_{a+b}$.

The action is needed because the configuration attached to a base point records masses at integer addresses relative to that base point. We read the coordinate $\lambda(z)$ at a base point u as the mass attached to the physical point $u - z$. If the base point is shifted from u to $u + a$, then the same physical point is written as

$$u - z = (u + a) - (z + a).$$

Thus the mass formerly stored at address z must, at the new base point $u + a$, be stored at address $z + a$. Equivalently, $\lambda_{\text{new}}(z + a) = \lambda_{\text{old}}(z)$, which is precisely the rule

$$(\rho_a \lambda)(z) = \lambda(z - a).$$

We need this relabeling because opposite faces of neighboring cubes represent the same physical points. The action ρ_a is what allows the local fillings on adjacent cubes to match along those shared boundary points and to assemble into one globally defined equivariant field. It will also yield the key identity $\varphi(x - z) = \Lambda(x)(z)$, from which the partition-of-unity and packing relations follow.

5.3. Constructing the equivariant configuration field Λ

We first construct the field on the closed fundamental cube

$$F = [0, 1]^d$$

and only afterwards extend it equivariantly to all of $\mathbb{R}^d$. For $i = 1, \dots, d$, write

$$F_i^- = \{x \in F : x_i = 0\}, \quad F_i^+ = \{x \in F : x_i = 1\} = F_i^- + e_i.$$

The map on the cube will satisfy the opposite-face compatibility relations

$$\Lambda_F(x + e_i) = \rho_{e_i} \Lambda_F(x), \quad x \in F_i^-. \quad (5.3)$$

These relations are precisely what is needed for the copies on neighboring integer cubes to fit together continuously.

We triangulate F by the standard staircase triangulation. For every permutation π of $1, \dots, d$, the corresponding d -simplex is

$$\text{conv}\{0, e_{\pi(1)}, e_{\pi(1)} + e_{\pi(2)}, \dots, e_{\pi(1)} + \dots + e_{\pi(d)}\}. \quad (5.4)$$

The triangulations induced on opposite faces of the cube agree after translation by the corresponding coordinate vector. In other words, when one translates a simplex lying in a face $x_i = 0$ by e_i , one obtains exactly the corresponding simplex in the opposite face $x_i = 1$.

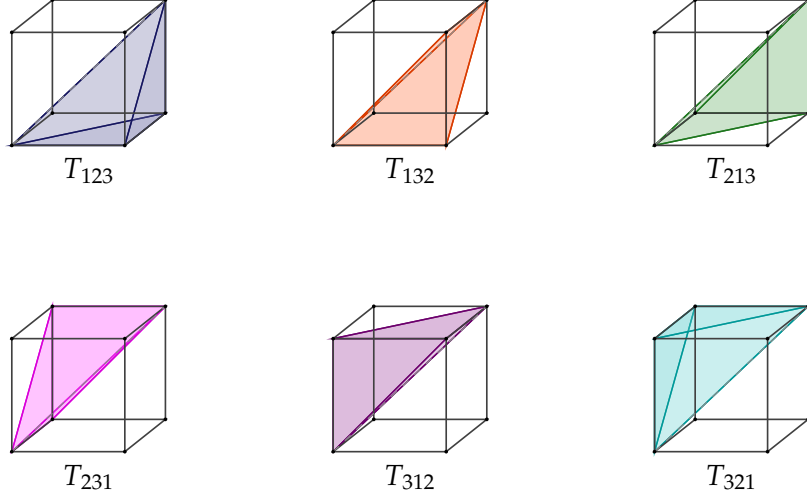


FIGURE 13. The standard staircase triangulation in dimension three. Each panel shows the unit cube from the same perspective and shades one of the six tetrahedra T_π . Every tetrahedron contains the long diagonal from $(0, 0, 0)$ to $(1, 1, 1)$; together the six shaded tetrahedra fill the cube and meet along common faces.

For a triangulated polyhedron, the j -skeleton is the union of all simplices of dimension at most j . In particular, the 0-skeleton of the triangulated cube consists precisely of its corners.

We begin on the 0-skeleton, that is, on the corners of the cube. Choose one hard configuration λ^0 . For every cube vertex $\varepsilon \in \{0, 1\}^d$, set

$$\Lambda_F(\varepsilon) = \rho_\varepsilon \lambda^0. \quad (5.5)$$

The commuting relations $\rho_a \rho_b = \rho_{a+b}$ show that these values already satisfy (5.3) at the corners, that is, on the 0-skeleton. For the concrete four-row, six-column illustration in dimension two, we take

$$\text{supp } \lambda^0 = \{(0, 0), (1, 1), (2, 2), (3, 3)\}.$$

Since $(\rho_a \lambda)(z) = \lambda(z - a)$, the support is translated by $+a$. Consequently,

$$\text{supp}(\rho_{e_1} \lambda^0) = \{(1, 0), (2, 1), (3, 2), (4, 3)\},$$

$$\text{supp}(\rho_{e_2} \lambda^0) = \{(0, 1), (1, 2), (2, 3), (3, 4)\},$$

$$\text{supp}(\rho_{e_1+e_2} \lambda^0) = \{(1, 1), (2, 2), (3, 3), (4, 4)\}.$$

For the explicit filling used in the later four-row, six-column example, the two non-corner values displayed below are

$$\Lambda(0, 0.5) = \delta_{(5,0)} + \delta_{(0,1)} + \frac{1}{2}\delta_{(1,2)} + \frac{1}{2}\delta_{(2,2)} + \delta_{(3,3)}, \quad (5.6)$$

$$\Lambda(0.9, 0.55) = \delta_{(6,0)} + \delta_{(1,1)} + \frac{4}{5}\delta_{(2,2)} + \frac{1}{5}\delta_{(3,2)} + \delta_{(4,3)}. \quad (5.7)$$

The occupied columns are pairwise distinct in each configuration.

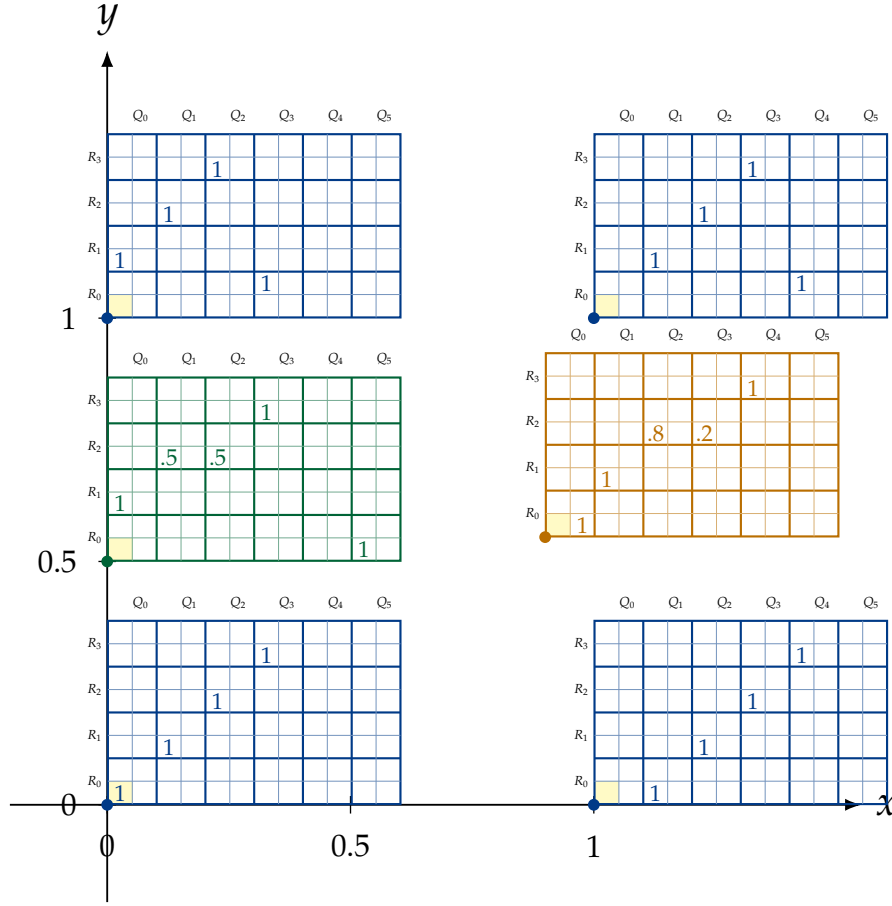


FIGURE 14. The legal configuration field Λ on one fundamental square for $T = \mathbb{Z} \times 4\mathbb{Z}$ and $P = 6\mathbb{Z} \times \mathbb{Z}$. The blue tables are the four equivariant corner values. The green and orange tables are the values in (5.6) and (5.7); in each table the positive entries occupy distinct columns. This is the same explicit field used in the second example in Subsection 6.2, whose tiling and buffered packing appear in Figure 19.

The first extension step already explains why interpolation through hard configurations alone is insufficient. Even on an edge, the straight segment between the endpoint hard configurations may leave the soft-configuration space: two rows can acquire positive mass in the same column. We therefore subdivide the edge, choose a polygonal path of soft configurations, and interpolate affinely along its segments. On an opposite edge, the path is defined by the appropriate translate ρ_{e_i} . This is the case $j = 1$ of Theorem 5.2.

We now fill the higher-dimensional simplices of F inductively. Suppose that Λ_F has already been defined on the $(j - 1)$ -skeleton and obeys the opposite-face relations there. A simplex in a boundary face cannot be filled independently of its translated copy on the opposite face. More precisely, if $\sigma \subset F_i^-$, then $\sigma + e_i \subset F_i^+$ must be filled by the translated rule $\Lambda_F(x + e_i) = \rho_{e_i}\Lambda_F(x)$. If σ lies in several coordinate faces, the same rule applies to every copy obtained by adding the corresponding sums of coordinate vectors. Thus the boundary identifications partition the finitely many j -simplices into finitely many classes. Interior simplices form singleton classes. We choose one representative from each class, fill only that representative, and obtain every other member of the class by equivariance. This guarantees compatibility on all identified faces by construction.

Fix one representative σ . Its boundary has already been filled and carries the simplexwise-affine map

$$f_\sigma = \Lambda_F|_{\partial\sigma} : \partial\sigma \longrightarrow C(T, P). \quad (5.8)$$

The boundary triangulation has finitely many vertices, and every vertex value has finite support, so only finitely many addresses occur there; call their union E_σ . Each row already occurs, since every boundary configuration has row sum one. Nevertheless, some row–column boxes may contain no address from E_σ . For each such missing box choose one arbitrary address and adjoin it to E_σ . The resulting family A_σ is finite and contains at least one address in every box. The boundary map still lands in the corresponding finite subcomplex $C_{m,n}(A_\sigma)$. We therefore apply the finite abstract Theorem 7.1 to this prescribed address system.

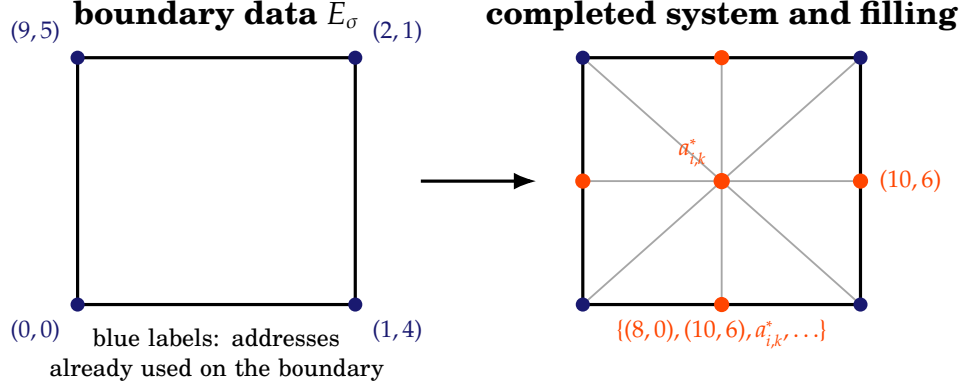


FIGURE 15. Finite bookkeeping before filling a two-dimensional parameter cell, shown schematically as a square. The boundary uses the finite address set E_σ shown in blue. We adjoin addresses from every missing row–column class and then triangulate the interior; new data are shown in orange. Rows are already nonempty, so the completion is needed for missing boxes, not for missing rows.

Since $j \leq d \leq n - m$, Theorem 7.1, applied to $C_{m,n}(A_\sigma)$, extends f_σ simplexwise affinely over σ , using only addresses from the prescribed finite system A_σ and leaving the existing boundary triangulation unchanged. If $\sigma + a$ is another simplex in the same opposite-face identification class, define its filling by

$$\Lambda_F(x + a) = \rho_a \Lambda_F(x), \quad x \in \sigma.$$

The relation $\rho_a \rho_b = \rho_{a+b}$ makes this prescription consistent on intersections of several faces. Adjacent fillings have the same vertex values on their common face, so their affine formulas agree there. Repeating this for $j = 1, \dots, d$ constructs a simplexwise affine map

$$\Lambda_F : F \longrightarrow C(T, P)$$

satisfying (5.3). Because F has only finitely many simplices after the construction, only finitely many addresses occur on the entire cube.

After the cube has been filled, we extend the map equivariantly to all of $\mathbb{R}^d$. Every $x \in \mathbb{R}^d$ has a unique representation

$$x = a + u, \quad a \in \mathbb{Z}^d, \quad u \in [0, 1)^d.$$

Define

$$\Lambda(x) = \rho_a \Lambda_F(u). \tag{5.9}$$

Then

$$\Lambda(x + b) = \rho_b \Lambda(x), \quad b \in \mathbb{Z}^d,$$

so the desired equivariance holds. It remains only to check continuity across the boundaries of adjacent cubes. If, for example, $u_i = 0$, the limit from the neighboring cube on the negative side is

$$\rho_{a-e_i} \Lambda_F(u + e_i) = \rho_{a-e_i} \rho_{e_i} \Lambda_F(u) = \rho_a \Lambda_F(u),$$

by (5.3). This equals the value in (5.9). The same calculation works at intersections of several coordinate faces. Hence Λ is continuous on all of $\mathbb{R}^d$ and is simplexwise affine on the translated cube triangulations.

5.4. The scalar partition of unity

The final scalar function is the coordinate of the zero address. We now show that it is a compactly supported piecewise-affine T -partition of unity:

$$\varphi(x) = \Lambda(x)(0), \quad x \in \mathbb{R}^d. \quad (5.10)$$

Since coordinate evaluation is continuous, φ is continuous. For every $z \in \mathbb{Z}^d$, equivariance gives the especially simple identity

$$\varphi(x - z) = \Lambda(x - z)(0) = (\rho_{-z}\Lambda(x))(0) = \Lambda(x)(z). \quad (5.11)$$

Thus the integer translates of the one scalar function are exactly the coordinates of the configuration $\Lambda(x)$.

Since $-T = T = R_0$, the row-normalization condition gives the partition-of-unity identity

$$\sum_{t \in T} \varphi(x + t) = \sum_{t \in T} \Lambda(x)(-t) = 1, \quad x \in \mathbb{R}^d. \quad (5.12)$$

If $p \in P \setminus \{0\}$ and both $\varphi(x)$ and $\varphi(x - p)$ were positive, then (5.11) implies that $\Lambda(x)$ is positive at the distinct addresses 0 and p , which lie in the same column $Q_0 = P$. This contradicts (4.7). Hence

$$\varphi(x)\varphi(x - p) = 0, \quad x \in \mathbb{R}^d, \quad p \in P \setminus \{0\}. \quad (5.13)$$

Only finitely many addresses occur while x ranges over one closed unit cube; denote their set by E . If $u \in [0, 1]^d$ and $\varphi(u - z) > 0$, then (5.11) implies $z \in E$. Therefore

$$\text{supp } \varphi \subset [0, 1]^d - E,$$

which is compact. Hence $\varphi \in C_c(\mathbb{R}^d)$. Since Λ is simplexwise affine, φ is piecewise affine on the translated periodic triangulation. Together with (5.12), this proves that φ is a compactly supported piecewise-affine T -partition of unity.

5.5. Thresholding

For fixed x , the nonzero terms $\varphi(x + t)$, $t \in T$, are the positive coordinates in one row of $\Lambda(x)$. Each of the other $m - 1$ rows must use at least one column, and these columns are distinct by the column-exclusion rule. The chosen row can therefore use at most

$$D = n - m + 1$$

columns. Hence at most D members of the T -partition of unity (5.12) are nonzero at any point, and one of them is at least $1/D$. Choose

$$0 < \tau < \frac{1}{D}$$

and set

$$U = \{x : \varphi(x) > \tau\}, \quad K = \{x : \varphi(x) \geq \tau\}.$$

Then U is open, K is compact, and $U + T = \mathbb{R}^d$. Moreover,

$$K \cap (K + p) = \emptyset, \quad p \in P \setminus \{0\},$$

by (5.13). Since K is compact and P is discrete,

$$\delta = \inf_{p \in P \setminus \{0\}} \text{dist}(K, K + p) > 0. \quad (5.14)$$

To see the strict positivity, choose $R > 0$ so large that $\text{dist}(K, K + p) > 1$ whenever $\|p\|_2 > R$. Only finitely many nonzero $p \in P$ satisfy $\|p\|_2 \leq R$, and each of the corresponding compact sets K and $K + p$ is disjoint. The minimum of their positive distances is therefore positive.

5.6. The first-hit selector

Let F_T be a bounded half-open fundamental parallelepiped for T and enumerate $T = \{t_1, t_2, \dots\}$. For $x \in F_T$, define

$$j(x) = \min\{j : x + t_j \in U\}.$$

This is defined because $U + T = \mathbb{R}^d$. Set

$$\Omega = \{x + t_{j(x)} : x \in F_T\}. \quad (5.15)$$

For each j , the set

$$E_j = \{x \in F_T : x + t_j \in U, x + t_i \notin U \text{ for } i < j\}$$

is Borel and

$$\Omega = \bigcup_j (E_j + t_j).$$

The function φ is affine on every simplex of a periodic triangulation and has compact support. It is therefore affine on the cells of a finite polyhedral subdivision of its support. Consequently, U is a finite union of relatively open polyhedral sets. Since both F_T and U are bounded, only finitely many translates $F_T + t_j$ can meet U . Hence only finitely many sets E_j are nonempty, and each nonempty E_j is a finite union of half-open polyhedral pieces. It follows that Ω is a finite union of bounded half-open polyhedral pieces.

The set Ω contains exactly one representative of every T -coset, so it is a T -fundamental domain. Moreover, $\Omega \subset U \subset K$, and therefore

$$\text{dist}(\Omega, \Omega + p) \geq \delta, \quad p \in P \setminus \{0\}.$$

For any $0 < \varepsilon < \delta/2$, the set $\Omega + B_\varepsilon$ packs with P . The inclusion $\Omega \subset K$ also proves boundedness.

Undoing the initial common linear normalization proves commensurable sufficiency. Indeed, if $S \in GL_d(\mathbb{R})$ is the normalizing map, then $S^{-1}(B_\varepsilon)$ contains a Euclidean ball of some positive radius. Thus a positive packing buffer in the normalized coordinates yields a positive Euclidean packing buffer for the original lattices.

6. EXAMPLES

We now record four explicit constructions. In each case we list the active addresses that occur in the construction. The first example is written out in detail, including the simplexwise formulas for Λ , the induced scalar function φ , and the resulting fundamental domain Ω .

6.1. A fully worked example: $T = \mathbb{Z}^2$ and $P = 3\mathbb{Z} \times \mathbb{Z}$

We now carry out the construction explicitly for

$$T = \mathbb{Z}^2, \quad P = 3\mathbb{Z} \times \mathbb{Z}.$$

Then

$$T + P = \mathbb{Z}^2, \quad m = [\mathbb{Z}^2 : T] = 1, \quad n = [\mathbb{Z}^2 : P] = 3,$$

so the sharp equality case $n = m + d$ holds when $d = 2$. The three columns are the congruence classes of the first coordinate modulo 3, and there is only one row.

We work on the unit square with vertices

$$v_0 = (0, 0), \quad v_1 = (1, 0), \quad v_2 = \left(1, \frac{1}{2}\right), \quad v_3 = (1, 1), \quad v_4 = (0, 1), \quad v_5 = \left(0, \frac{1}{2}\right),$$

and triangulate it by

$$\Delta_1 = [v_0, v_1, v_5], \quad \Delta_2 = [v_1, v_2, v_5], \quad \Delta_3 = [v_2, v_3, v_5], \quad \Delta_4 = [v_3, v_4, v_5].$$

Assign the hard configurations

$$v_0 \mapsto \delta_{(0,0)}, \quad v_1 \mapsto \delta_{(1,0)}, \quad v_2 \mapsto \delta_{(0,0)}, \quad v_3 \mapsto \delta_{(1,1)}, \quad v_4 \mapsto \delta_{(0,1)}, \quad v_5 \mapsto \delta_{(-1,0)}.$$

On each triangle we interpolate affinely. Relative to $[0, 1]^2$, we have

$$\begin{aligned} \Delta_1 &= \{x + 2y \leq 1\}, & \Delta_2 &= \{y \leq \tfrac{1}{2}, x + 2y \geq 1\}, \\ \Delta_3 &= \{y \geq \tfrac{1}{2}, 2y \leq 1 + x\}, & \Delta_4 &= \{2y \geq 1 + x\}, \end{aligned}$$

and obtain the configuration field

$$\Lambda(x, y) = \begin{cases} (1 - x - 2y)\delta_{(0,0)} + x\delta_{(1,0)} + 2y\delta_{(-1,0)}, & (x, y) \in \Delta_1, \\ (x + 2y - 1)\delta_{(0,0)} + (1 - 2y)\delta_{(1,0)} + (1 - x)\delta_{(-1,0)}, & (x, y) \in \Delta_2, \\ (x - 2y + 1)\delta_{(0,0)} + (2y - 1)\delta_{(1,1)} + (1 - x)\delta_{(-1,0)}, & (x, y) \in \Delta_3, \\ (2y - 1 - x)\delta_{(0,1)} + x\delta_{(1,1)} + (2 - 2y)\delta_{(-1,0)}, & (x, y) \in \Delta_4. \end{cases} \quad (6.1)$$

These formulas agree on common edges. Moreover,

$$\Lambda(1, y) = \rho_{(1,0)}\Lambda(0, y), \quad \Lambda(x, 1) = \rho_{(0,1)}\Lambda(x, 0),$$

so Λ extends to all of $\mathbb{R}^2$ by

$$\Lambda(u + z) = \rho_z\Lambda(u), \quad u \in [0, 1]^2, \quad z \in \mathbb{Z}^2.$$

The associated scalar function is determined by

$$\varphi(u - z) = \Lambda(u)(z), \quad u \in [0, 1]^2, \quad z \in \mathbb{Z}^2.$$

Only the five addresses

$$(0, 0), (1, 0), (-1, 0), (0, 1), (1, 1)$$

occur. On the fundamental square the nonzero coordinate functions are

z	Δ_1	Δ_2	Δ_3	Δ_4
$(0, 0)$	$1 - x - 2y$	$x + 2y - 1$	$x - 2y + 1$	0
$(1, 0)$	x	$1 - 2y$	0	0
$(-1, 0)$	$2y$	$1 - x$	$1 - x$	$2 - 2y$
$(0, 1)$	0	0	0	$2y - 1 - x$
$(1, 1)$	0	0	$2y - 1$	x .

(6.2)

Hence φ is a nonnegative compactly supported continuous piecewise-affine function. By construction,

$$\sum_{t \in \mathbb{Z}^2} \varphi(x + t) = 1, \quad \varphi(x)\varphi(x - p) = 0 \quad (p \in (3\mathbb{Z} \times \mathbb{Z}) \setminus \{0\}).$$

Choose $\tau = 0.3$. In this detailed example and in Figure 16, the first-hit selector uses the active addresses $((0, 0), (1, 0), (-1, 0), (0, 1), (1, 1))$, equivalently the enumeration

$$(t_1, t_2, t_3, t_4, t_5) = ((0, 0), (-1, 0), (1, 0), (0, -1), (-1, -1)) \quad (6.3)$$

of the corresponding T -translates, where $t_j = -a_j$. For $u \in [0, 1]^2$, let $j(u)$ be the first index for which $\Lambda(u)(a_j) > \tau$. Since on every triangle three nonnegative coefficients add to 1, some coefficient is at least $1/3$, so $j(u)$ is well defined. We then set

$$\Omega = \{u - a_{j(u)} : u \in [0, 1]^2\}. \quad (6.4)$$

Equivalently, if

$$E_j = \{u \in [0, 1]^2 : \Lambda(u)(a_j) > \tau \text{ and } \Lambda(u)(a_i) \leq \tau \text{ for } i < j\},$$

then

$$\Omega = \bigcup_{j=1}^5 (E_j - a_j).$$

Thus Ω is a finite union of bounded half-open polygons and a $\mathbb{Z}^2$ -fundamental domain. Since $\Omega \subset \{\varphi \geq 0.3\}$ and the latter set has positive distance from all its nontrivial translates by $3\mathbb{Z} \times \mathbb{Z}$, the set $\Omega + B_\varepsilon$ packs with $3\mathbb{Z} \times \mathbb{Z}$ for all sufficiently small $\varepsilon > 0$.

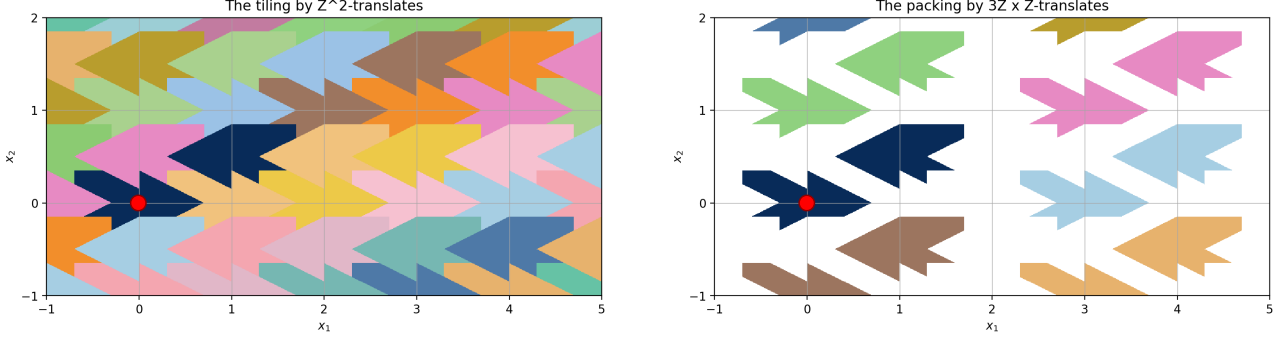


FIGURE 16. The domain obtained from the first-hit enumeration (6.3). Left: the tiling $\Omega + \mathbb{Z}^2$ on $[-1, 5] \times [-1, 2]$. Right: the corresponding translates by $3\mathbb{Z} \times \mathbb{Z}$ on the same rectangle. The unshifted domain is dark blue, and the visible gaps give the positive packing buffer.

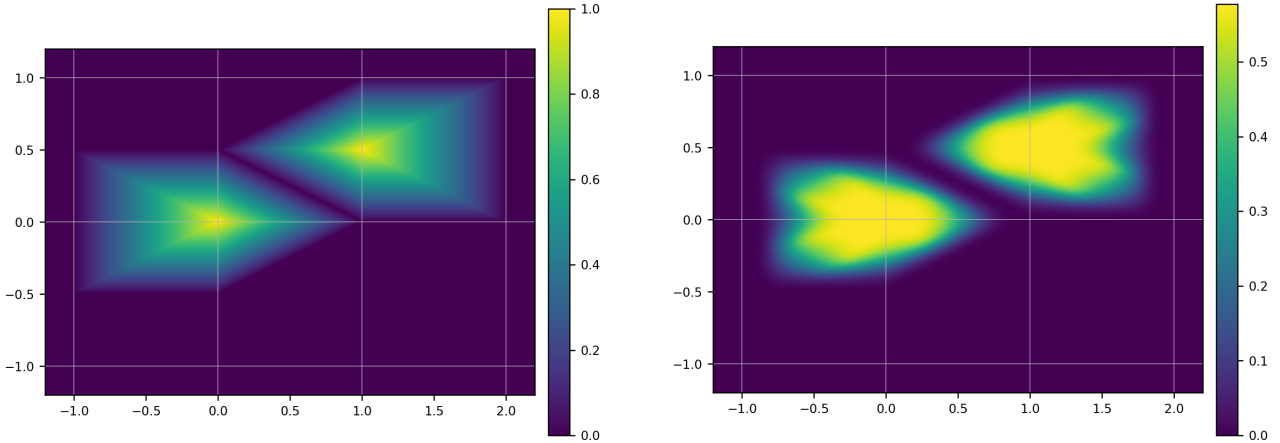


FIGURE 17. Left: a two-dimensional color plot of the compactly supported continuous piecewise-affine function φ produced from the soft-configuration field. Right: a two-dimensional color plot of the smooth window g obtained in Part B by cutting φ below a positive level, mollifying the result, and then normalizing the periodic energy.

6.2. Two further equality-case examples

We record two explicit constructions at the planar equality threshold. For $T = 7\mathbb{Z} \times \mathbb{Z}$ and $P = 3\mathbb{Z} \times 3\mathbb{Z}$, one has $m = 7$ and $n = 9 = m + 2$; the positive addresses used by the construction are

$$\begin{aligned} E_{7,9} = \{ & (1, 1), (1, 0), (0, 1), (0, 0), (-1, 1), (-1, 0), (-2, 1), \\ & (-2, 0), (-2, -1), (-3, 0), (-3, -1), (-4, 0), (-4, -1), \\ & (-5, 0), (-5, -1), (-5, -2), (-6, -1), (-6, -2), (-7, -2) \}. \end{aligned} \quad (6.5)$$

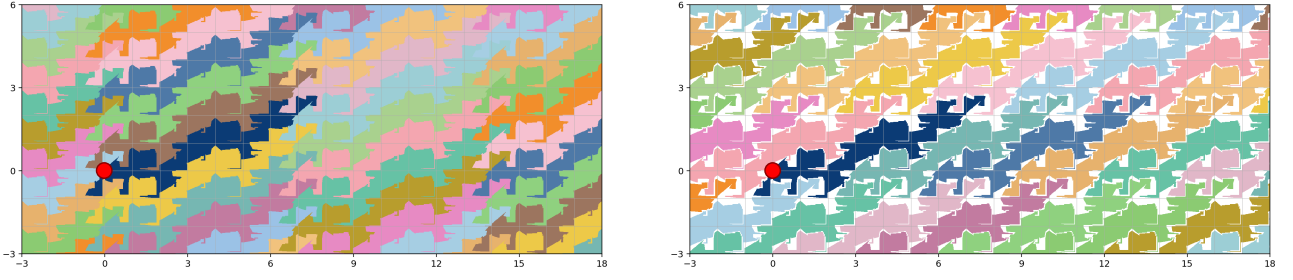


FIGURE 18. The nineteen-address construction for $T = 7\mathbb{Z} \times \mathbb{Z}$ and $P = 3\mathbb{Z} \times 3\mathbb{Z}$. From left to right: the T -tiling and the buffered P -packing.

For $T = \mathbb{Z} \times 4\mathbb{Z}$ and $P = 6\mathbb{Z} \times \mathbb{Z}$, one has $m = 4$ and $n = 6 = m + 2$; the positive addresses are

$$E_{4,6} = \{(0,0), (1,0), (2,1), (3,2), (4,3), (5,0), (6,0), (0,1), (1,1), (1,2), (2,2), (2,3), (3,3), (3,4), (4,4)\}. \quad (6.6)$$

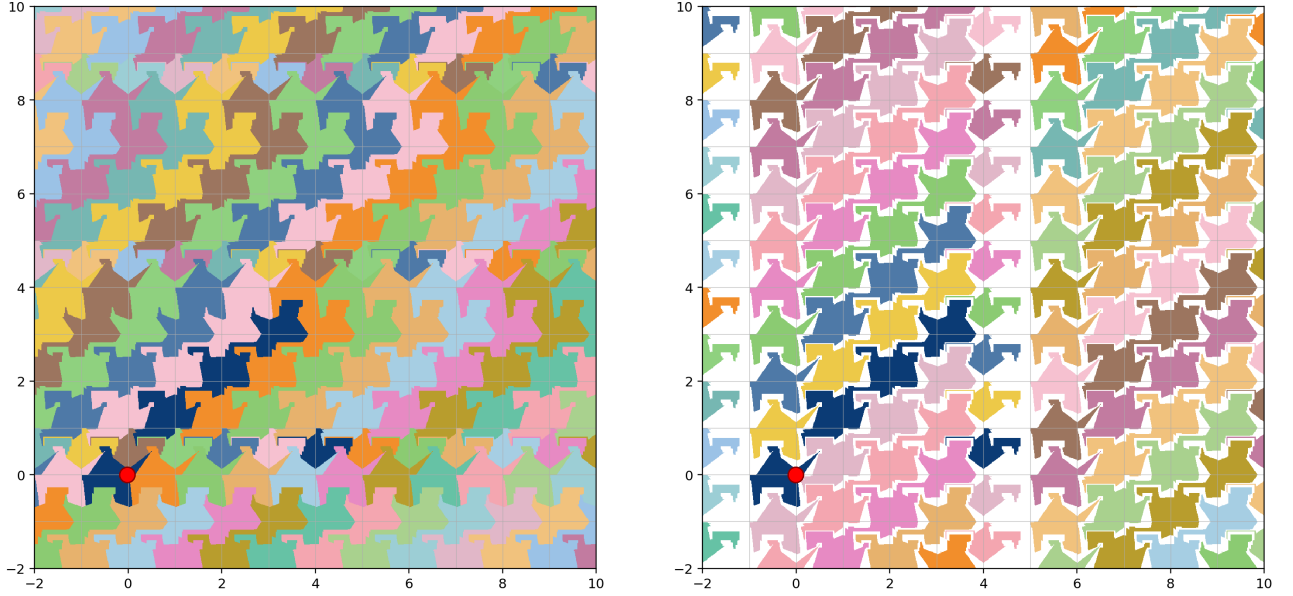


FIGURE 19. The fifteen-address construction for $T = \mathbb{Z} \times 4\mathbb{Z}$ and $P = 6\mathbb{Z} \times \mathbb{Z}$. From left to right: the T -tiling and the buffered P -packing.

7. ABSTRACT ROW-COLUMN COMPLEXES AND PROOF OF THEOREM 5.2

The extension problem used in Section 5 depends only on the row-column incidence structure, not on the ambient lattice groups. We now forget the group origin of the address classes and prove a stronger abstract theorem.

Fix integers $1 \leq m \leq n$. For every row $i = 1, \dots, m$ and column $j = 1, \dots, n$, let $A_{i,j}$ be a nonempty set, not necessarily finite. The sets are regarded as disjoint labeled sets, and

$$A_i = \bigsqcup_{j=1}^n A_{i,j}.$$

An abstract soft configuration is a tuple $\lambda = (\lambda_1, \dots, \lambda_m)$ such that every λ_i is a finitely supported probability vector on A_i and distinct positive coordinates never use the same column. Explicitly,

$$\lambda_i(a) \geq 0, \quad \#\text{supp } \lambda_i < \infty, \quad \sum_{a \in A_i} \lambda_i(a) = 1,$$

and, whenever $a \in A_{i,j}$ and $b \in A_{i',j}$ are distinct labeled addresses,

$$\lambda_i(a)\lambda_{i'}(b) = 0.$$

The resulting, possibly infinite, polyhedral complex is denoted by $C_{m,n}(A)$. We regard it as a union of finite-dimensional cells inside the coordinate space

$$\prod_{i=1}^m \mathbb{R}^{(A_i)},$$

where $\mathbb{R}^{(A_i)}$ denotes the finitely supported real functions on A_i . If $S = (S_1, \dots, S_m)$ is an admissible finite support pattern, its cell is

$$C_S = \prod_{i=1}^m \Delta(S_i), \quad C_{m,n}(A) = \bigcup_{S \text{ admissible}} C_S.$$

When every $A_{i,j}$ is finite, this is the finite polyhedral complex used in the proof below. Simplexwise affine maps into $C_{m,n}(A)$ are defined by the same vertex-interpolation rule as in Section 4.

Theorem 7.1. *Let every $A_{i,j}$ be a nonempty set, finite or infinite, and let $C_{m,n}(A)$ consist of the finitely supported abstract soft configurations defined above. For every integer*

$$1 \leq k \leq n - m,$$

every simplexwise affine map on a finitely triangulated S^{k-1} extends simplexwise affinely over a finitely triangulated B^k without changing the boundary triangulation or its vertex values. Moreover, for each boundary map there are finite nonempty subsets

$$B_{i,j} \subset A_{i,j}$$

containing every address that occurs in a boundary vertex value, and the extension may be chosen to take values in the finite subcomplex $C_{m,n}(B) \subset C_{m,n}(A)$. If $n = m$, the space $C_{m,m}(A)$ is nonempty.

Remark 7.2 (A topological claim suggested by ChatGPT). In the finite-address setting, ChatGPT claims that every continuous map $f : S^{j-1} \rightarrow C_{m,n}(A)$ extends continuously over B^j whenever $1 \leq j \leq n - m$, and hence that $C_{m,n}(A)$ is $(n - m - 1)$ -connected. An independent verification of this stronger continuous statement goes beyond the scope of the present paper. It is not used below; the proved simplexwise-affine extension theorem is sufficient for all constructions in the paper.

7.1. The finite-address reduction

We first show that every finite simplexwise-affine problem lives inside a finite subcomplex, even when the ambient address classes are infinite.

Lemma 7.3. *Let every $A_{i,j}$ be nonempty, possibly infinite. Let K be a finite simplicial complex and let*

$$f : |K| \longrightarrow C_{m,n}(A)$$

be simplexwise affine. Then there are finite nonempty sets $B_{i,j} \subset A_{i,j}$ such that

$$f(|K|) \subseteq C_{m,n}(B).$$

The sets $B_{i,j}$ may be chosen to contain every address occurring in the support of a vertex value $f(v)$, $v \in K^{(0)}$.

Proof. For every row and column put

$$E_{i,j} = \bigcup_{v \in K^{(0)}} (\text{supp } f(v)_i \cap A_{i,j}).$$

The triangulation has finitely many vertices, and each vertex value is a finitely supported configuration. Hence every $E_{i,j}$ is finite. If $E_{i,j}$ is nonempty, set $B_{i,j} = E_{i,j}$; if it is empty, choose one address $b_{i,j} \in A_{i,j}$ and set $B_{i,j} = \{b_{i,j}\}$. Thus B is a finite complete row-column subsystem.

Let $\sigma = [v_0, \dots, v_p]$ be a simplex of K and write $x = \sum_{\ell=0}^p t_\ell v_\ell$ in barycentric coordinates. Since f is affine on σ ,

$$f(x) = \sum_{\ell=0}^p t_\ell f(v_\ell).$$

Every coordinate outside the union of the supports of the vertex values is therefore zero. All those supports are contained in the sets $B_{i,j}$, so $f(\sigma) \subseteq C_{m,n}(B)$. Taking the union over the finitely many simplices proves the assertion. $\square$

This is a finite-address reduction, not a local-finiteness assertion about the entire infinite complex. Indeed, when some $A_{i,j}$ is infinite, infinitely many cells can meet at one hard configuration. What is used here is only that a simplexwise affine problem on a finite triangulation activates finitely many addresses.

To prove Theorem 7.1, it now suffices to establish the theorem when all $A_{i,j}$ are finite. Once that finite case is known, Lemma 7.3 places any boundary map in a finite complete subcomplex $C_{m,n}(B)$, and its extension there is also an extension in $C_{m,n}(A)$. For the finite-case proof below, we therefore assume that every $A_{i,j}$ is finite. This assumption is used in the last-column decomposition to make the cover by the sets U_v finite.

7.2. A finite-piece extension lemma

The next lemma is a convenient device for the last-column decomposition. All polyhedral subcomplexes below are unions of cells of one fixed finite polyhedral structure.

Lemma 7.4. *Let I be a finite nonempty index set and let*

$$X = \bigcup_{\alpha \in I} X_\alpha$$

be a finite polyhedral complex written as a union of subcomplexes. Suppose that a nonempty subcomplex A satisfies

$$A \subseteq X_\alpha, \quad \alpha \in I,$$

and

$$X_\alpha \cap X_\beta = A, \quad \alpha, \beta \in I, \quad \alpha \neq \beta.$$

Fix $k \geq 1$. Assume that, for every $\alpha \in I$ and every $0 \leq p \leq k$, every simplexwise affine map from a triangulated S^p into X_α extends simplexwise affinely over B^{p+1} without changing the boundary triangulation. Assume also that the same statement holds with A as target for every $0 \leq p \leq k-1$. Then every simplexwise affine map from a triangulated S^k into X extends simplexwise affinely over B^{k+1} .

Proof. Choose a finite triangulation K of S^k with respect to which the boundary map $f : |K| \rightarrow X$ is simplexwise affine. Thus f is affine on every simplex of K , and the image of each simplex lies in one cell of the fixed polyhedral structure on X . Choose $a_0 \in A$, add an interior vertex c , and identify the cone $c * K$ with B^{k+1} . Set $F(c) = a_0$. For every simplex $\sigma \in K$ put

$$\widehat{\sigma} = \text{conv}(\{c\} \cup \sigma).$$

We fill these conical simplices in increasing order of $\dim \sigma$; every filling is allowed to subdivide the interior of $\widehat{\sigma}$ but must leave its already triangulated boundary unchanged.

Let C_σ be the unique smallest target cell containing $f(\sigma)$. If $C_\sigma \not\subseteq A$, then it is contained in a unique piece $X_{\alpha(\sigma)}$: if it belonged to two distinct pieces, it would be contained in their intersection A . Fix $\alpha_0 \in I$ and define

$$Z_\sigma = \begin{cases} X_{\alpha(\sigma)}, & C_\sigma \not\subseteq A, \\ A, & C_\sigma \subseteq A \text{ and } \dim \sigma < k, \\ X_{\alpha_0}, & C_\sigma \subseteq A \text{ and } \dim \sigma = k. \end{cases}$$

The assignments are nested along faces:

$$\eta \subsetneq \sigma \implies Z_\eta \subseteq Z_\sigma. \quad (7.1)$$

Indeed, $f(\eta) \subseteq f(\sigma)$ implies that the smallest cell C_η is a face of C_σ . If $C_\sigma \not\subseteq A$, both cells lie in the same unique piece unless $C_\eta \subseteq A$, in which case $Z_\eta = A \subset X_{\alpha(\sigma)}$. If $C_\sigma \subseteq A$, then $C_\eta \subseteq A$ as well.

Suppose that the regions $\widehat{\eta}$ have already been filled for all proper faces η of σ . The top face σ maps into Z_σ , and each side face $\widehat{\eta}$ maps into $Z_\eta \subseteq Z_\sigma$. Thus the complete triangulated boundary of the $(\dim \sigma + 1)$ -simplex $\widehat{\sigma}$ maps into Z_σ . If $Z_\sigma = A$, then $\dim \sigma \leq k - 1$, so the extension hypothesis for A applies with $p = \dim \sigma$. Otherwise the hypothesis for the appropriate piece X_α applies, since $\dim \sigma \leq k$. We therefore triangulate $\widehat{\sigma}$ relative to its boundary and fill it inside Z_σ .

Proceeding through all simplices of K fills the cone $c * K$. Whenever two filled regions meet, their common face had already been triangulated and assigned vertex values before either interior was filled. The affine formulas therefore agree on that face. Hence the resulting map is simplexwise affine and continuous on B^{k+1} . $\square$

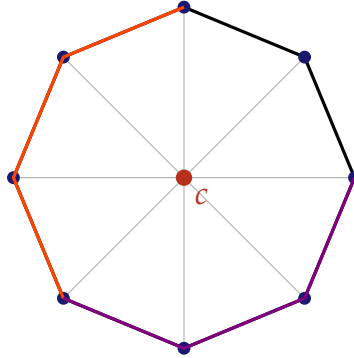


FIGURE 20. The ball is filled from the interior vertex toward the boundary, processing the regions associated with lower-dimensional boundary simplices before those associated with higher-dimensional ones.

7.3. The last-column decomposition

Assume throughout this subsection that $n > m$. Let

$$A_{\bullet,n} = \bigsqcup_{i=1}^m A_{i,n}$$

be the labeled addresses in the last column. For $v \in A_{\bullet,n}$, define

$$U_v = \{\lambda \in C_{m,n}(A) : \lambda_i(a) = 0 \text{ for every } i \text{ and every } a \in A_{i,n} \setminus \{v\}\}.$$

Thus the last column is either unused, or v is its only positive address. Also put

$$A_0 = \{\lambda \in C_{m,n}(A) : \lambda_i(a) = 0 \text{ for every } i \text{ and every } a \in A_{i,n}\}.$$

Every configuration either leaves the last column unused or uses a unique address v there. Consequently,

$$C_{m,n}(A) = \bigcup_{v \in A_{\bullet,n}} U_v, \quad U_v \cap U_w = A_0, \quad v, w \in A_{\bullet,n}, \quad v \neq w. \quad (7.2)$$

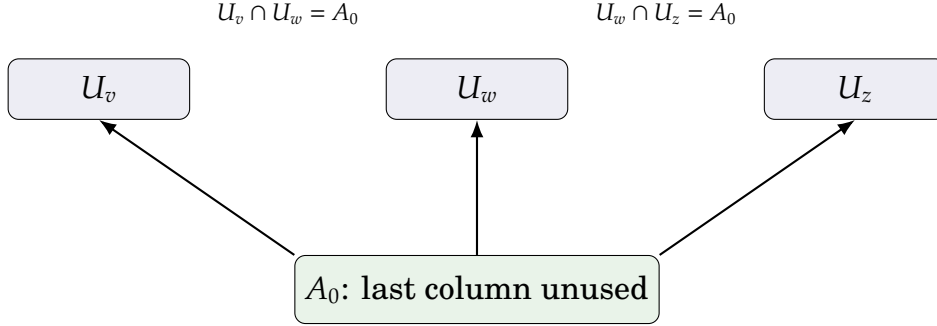


FIGURE 21. The last-column decomposition. Each piece permits at most one specified address in the last column, and all distinct pieces meet in the common subspace A_0 where that column is unused.

We now define the reduced address families explicitly. Deleting the last column gives

$$A^- = (A_{i,j}^-)_{1 \leq i \leq m, 1 \leq j \leq n-1}, \quad A_{i,j}^- = A_{i,j}.$$

Consequently,

$$A_0 \cong C_{m,n-1}(A^-). \quad (7.3)$$

If $v \in A_{i_0,n}$, let

$$\iota_v : \{1, \dots, m-1\} \longrightarrow \{1, \dots, m\} \setminus \{i_0\}$$

be the increasing bijection, and define

$$A_{\ell,j}^v = A_{\iota_v(\ell),j}, \quad 1 \leq \ell \leq m-1, \quad 1 \leq j \leq n-1.$$

Thus A^v is obtained by deleting row i_0 and column n and relabeling the remaining rows.

Fix $v \in A_{i_0,n}$. Define

$$(P_v \lambda)_{i_0}(v) = 1, \quad (P_v \lambda)_{i_0}(a) = 0, \quad a \in A_{i_0} \setminus \{v\},$$

and set $(P_v \lambda)_i = \lambda_i$ for $i \neq i_0$. Let L be the linear map that leaves every row other than i_0 unchanged and replaces the i_0 -th row by zero, and let e_v be the vector whose only nonzero coordinate is the value 1 at v . Then

$$P_v \lambda = L \lambda + e_v.$$

Thus P_v is the restriction of an affine map on the ambient coordinate space and is therefore continuous. It leaves the other rows unchanged and replaces the i_0 -th row by the fixed point mass at v . Its image is

$$Y_v = \{\lambda \in U_v : \lambda_{i_0}(v) = 1\}.$$

For $m \geq 2$, deleting the fixed row and the unused last column gives

$$Y_v \cong C_{m-1,n-1}(A^v), \quad (7.4)$$

while for $m = 1$ the set Y_v is one point.

The vertexwise definition makes compositions transparent. Fix $v \in A_{i_0,n}$, and suppose that f is affine on a simplex $\sigma = [v_0, \dots, v_p]$ and that

$$f(v_\ell) \in U_v, \quad \ell = 0, \dots, p.$$

In the application below, the whole map f takes values in U_v . If $x = \sum_{\ell=0}^p t_\ell v_\ell$ is written in barycentric coordinates, then, for every row i and every address $a \in A_{i,n} \setminus \{v\}$,

$$f(x)_i(a) = \sum_{\ell=0}^p t_\ell f(v_\ell)_i(a) = 0.$$

Hence $f(\sigma) \subset U_v$. Let C_S be the smallest soft-configuration cell containing the vertex values. It follows that the support pattern S uses no last-column address other than possibly v . The map P_v is affine on C_S and sends it into the cell obtained by replacing the support of row i_0 by $\{v\}$. Therefore,

$$P_v \left(\sum_{\ell=0}^p t_\ell f(v_\ell) \right) = \sum_{\ell=0}^p t_\ell P_v(f(v_\ell)).$$

Thus $P_v \circ f$ is simplexwise affine on the same triangulation; no target subdivision is required.

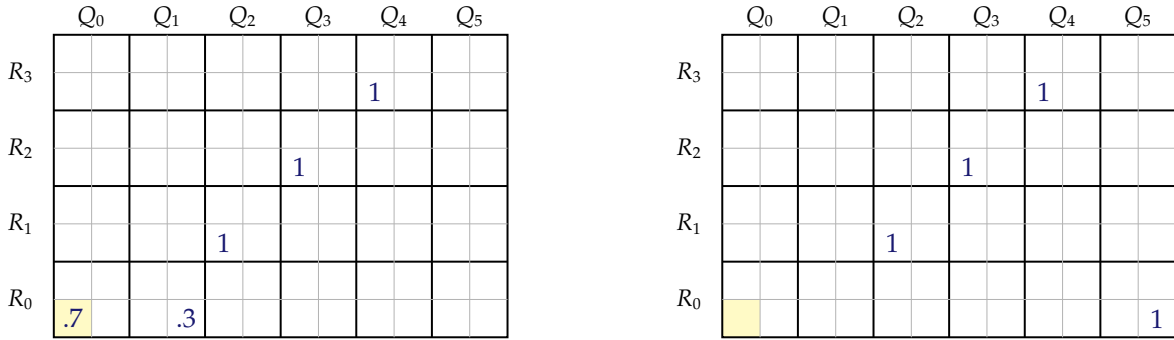


FIGURE 22. The affine map P_v in one example. On the left, the last column is unused and the first row splits its mass as $0.7 + 0.3$. On the right, P_v transfers the whole first-row mass to the chosen address $v \in A_{1,6}$ and leaves the other rows unchanged.

Proposition 7.5. *Fix $v \in A_{i_0,n}$. If $m = 1$, every simplexwise affine map $S^p \rightarrow U_v$ extends simplexwise affinely over B^{p+1} for every $p \geq 0$. If $m \geq 2$ and Theorem 7.1 is known for $C_{m-1,n-1}(A^v)$, the same conclusion holds for $0 \leq p \leq n - m - 1$.*

Proof. Let K triangulate S^p and let $f : |K| \rightarrow U_v$ be simplexwise affine. Take an outer copy of K and a concentric inner copy. For each p' -simplex $\sigma = [z_0, \dots, z_{p'}]$ of K , triangulate the region between its two copies, identified with $\sigma \times [0, 1]$, by the standard staircase simplices

$$\text{conv}\{(z_0, 0), \dots, (z_i, 0), (z_i, 1), \dots, (z_{p'}, 1)\}, \quad i = 0, \dots, p'.$$

Assign the outer vertex $(z, 0)$ the value $f(z)$ and the inner vertex $(z, 1)$ the value $P_v(f(z))$.

For a fixed σ , all outer and inner vertex values lie in one common cell: start with a cell containing the outer values and replace the support of row i_0 by its union with $\{v\}$. This remains admissible because U_v permits no other positive last-column address. Affine interpolation on every staircase simplex therefore stays in U_v . The triangulations agree on common prism faces, and the formulas there use the same vertex values, so the resulting map on the region between the two spheres is continuous. It equals f on the outer sphere and $P_v \circ f$ on the inner sphere.

If $m = 1$, the inner map is constant and fills the inner ball. If $m \geq 2$, then

$$p \leq n - m - 1 = (n - 1) - (m - 1) - 1,$$

so the induction hypothesis fills the inner map inside $Y_v \cong C_{m-1,n-1}(A^v)$. The inner filling keeps its boundary triangulation and vertex values, hence it joins the prism interpolation

without any further compatibility argument. Together the two parts form a simplexwise affine filling of f over a ball. $\square$

7.4. Proof of Theorem 7.1

All ingredients are now in place, and we combine them in an induction on the total number of rows and columns.

Proof. By Lemma 7.3, it is enough to prove the finite-address case. We now prove that case by induction on $m + n$. If $n = m$, choose one address $a_i \in A_{i,i}$ in every row and put weight one at a_i . The chosen columns are distinct, so $C_{m,m}(A)$ is nonempty.

Assume $n > m$ and that the finite-address theorem is known whenever the sum of the row and column numbers is smaller. Let K triangulate S^k , where $0 \leq k \leq n - m - 1$, and let $f : |K| \rightarrow C_{m,n}(A)$ be simplexwise affine. Every piece U_v has the required filling property through dimension $n - m - 1$ by Proposition 7.5, and $A_0 \cong C_{m,n-1}(A^-)$.

If $k = 0$, write the two values of f as x and y . Choose pieces U_v and U_w containing them. Since $n - 1 \geq m$, the set A_0 is nonempty; choose $a_0 \in A_0$. Proposition 7.5 gives a polygonal path from x to a_0 inside U_v and another from a_0 to y inside U_w . Their concatenation fills S^0 over B^1 .

Now let $k \geq 1$. For every $v \in A_{\bullet,n}$ and every $0 \leq p \leq k$, maps into U_v extend. For $0 \leq p \leq k - 1$,

$$p \leq k - 1 \leq n - m - 2 = (n - 1) - m - 1,$$

so the induction hypothesis gives the corresponding extensions into A_0 . The finite-piece Lemma 7.4, applied to (7.2), fills f over B^{k+1} . Since $j = k + 1$ ranges from 1 to $n - m$, this proves the finite-address case. The general case follows from Lemma 7.3, and the extension produced there lies in the finite subcomplex $C_{m,n}(B)$ asserted in the theorem. $\square$

Under the hypothesis $n \geq m + d$, apply Theorem 7.1 to the possibly infinite address sets

$$A_{i+1,j+1} = \mathcal{A}_{i,j}, \quad 0 \leq i \leq m - 1, \quad 0 \leq j \leq n - 1,$$

and restrict to $1 \leq k \leq d$. This gives Theorem 5.2. Its finite-address conclusion is precisely the conclusion of Lemma 7.3 in the lattice setting.

8. THE FULLY DENSE CASE: PARTITIONS OF UNITY AND SLOTS

Assume throughout this section that

$$\overline{T + P} = \mathbb{R}^d, \quad \text{covol}(T) < \text{covol}(P). \quad (8.1)$$

The group $T + P$ is countable; “fully dense” refers to the closure in (8.1). In this case there is no discrete quotient and hence no parameter torus on which a configuration field must be filled. The construction reduces to two elementary finite objects in the connected Euclidean space: small functions forming a periodic partition of unity and separated open sets into which their supports can be moved.

The geometric idea is the packet–slot matching used by Grepstad, Kolountzakis, and Spyridakis [24]; related common-domain constructions appear in [23]. At the level of characteristic functions, one cuts a source fundamental domain into many small pieces and assigns each piece wholly to a distinct target slot. This is the connected analogue of a hard configuration: every source packet chooses one slot, and no slot is used twice. We keep the same counting and density mechanism, but soften the source side by replacing characteristic functions with nonnegative elements of a partition of unity. The argument below is self-contained and does not import an existence theorem from those papers.

We develop the two finite objects in a form that will be reused in Section 9. Let V be an ℓ -dimensional Euclidean space and let $T_0, P_0 \subset V$ be full-rank lattices. A finite family

$h_1, \dots, h_N$ is a *finite T_0 -partition of unity* if

$$\sum_{i=1}^N \sum_{\tau \in T_0} h_i(v + \tau) = 1, \quad v \in V.$$

The functions h_i are called the partition elements. A *target slot family* is a finite family of bounded open sets $\widetilde{S}_j$, $j \in \{1, \dots, M\}$, such that $\widetilde{S}_j \cap (\widetilde{S}_{j'} + p_0) = \emptyset$ for all $j, j' \in \{1, \dots, M\}$ and $p_0 \in P_0$, unless $j = j'$ and $p_0 = 0$. A partition element fits slot j with margin $\eta > 0$ if a translate of $\text{supp } h_i + B_\eta^V$ is contained in $\widetilde{S}_j$, where

$$B_\eta^V = \{v \in V : \|v\|_2 < \eta\}.$$

The construction will be carried out in either of two concrete vector spaces. Since a nonzero globally affine function cannot have compact support, the “affine” alternative means continuous compactly supported piecewise-affine functions. We write $C_{c, \text{PA}}(V)$ for the functions that are affine on the cells of a finite polyhedral subdivision; the second choice will be $C_c^\infty(V)$.

8.1. Piecewise-affine and smooth partitions of unity

We first prove the required fine partition-of-unity statement separately in the two regularity classes.

Lemma 8.1 (Piecewise-affine lattice partition of unity). *For every full-rank lattice $T_0 \subset V$, every $\lambda_1 > 1$, and all sufficiently large integers s , there are $N = s^\ell$ nonzero functions*

$$h_1, \dots, h_N \in C_{c, \text{PA}}(V), \quad h_i \geq 0,$$

whose periodizations define a T_0 -periodic continuous piecewise-affine map

$$H = (H_1, \dots, H_N) : V \longrightarrow \Delta^{N-1}, \quad H_i(v) = \sum_{\tau \in T_0} h_i(v + \tau),$$

and satisfy

$$\sum_{i=1}^N H_i(v) = 1. \tag{8.2}$$

After a fixed linear normalization carrying T_0 to $\mathbb{Z}^\ell$, every $\text{supp } h_i$ is contained in a cube of side length less than λ_1/s .

Proof. Apply a linear map carrying T_0 to $\mathbb{Z}^\ell$. Partition the half-open unit cube into

$$C_a = \frac{1}{s}(a + [0, 1)^\ell), \quad a \in \{0, \dots, s-1\}^\ell.$$

Choose

$$0 < r < \frac{\lambda_1 - 1}{2s}.$$

On the torus $\mathbb{R}^\ell / \mathbb{Z}^\ell$, choose one point in the interior of every C_a and triangulate so that all these points are vertices and every closed vertex star has diameter less than r . This is obtained by starting from a triangulation containing the finitely many prescribed points and taking sufficiently many subdivisions. Lift it to a $\mathbb{Z}^\ell$ -periodic triangulation of $\mathbb{R}^\ell$.

Let ϑ_v be the nodal hat function at a vertex v of the lifted triangulation. Choose one representative of every vertex orbit modulo $\mathbb{Z}^\ell$. First assign the prescribed interior vertex of each C_a to that cube; assign every remaining representative to the unique half-open cube containing it. Define

$$h_a = \sum_{v \text{ assigned to } C_a} \vartheta_v, \quad H_a(x) = \sum_{q \in \mathbb{Z}^\ell} h_a(x + q).$$

Every h_a is nonzero, nonnegative, compactly supported, and piecewise affine. The periodic triangulation makes each H_a periodic and affine on every simplex. The nodal hat functions over all vertices of the periodic triangulation sum to one. Since every vertex orbit was assigned exactly once,

$$\sum_a H_a(x) = \sum_a \sum_{q \in \mathbb{Z}^\ell} h_a(x + q) = 1.$$

Thus $H = (H_a)_a$ maps into the probability simplex. Moreover, the star of a vertex assigned to C_a lies in $C_a + B_r$, so

$$\text{supp } h_a \subset C_a + B_r,$$

which is contained in a cube of side length $1/s + 2r < \lambda_1/s$. Undo the initial linear normalization. $\square$

Lemma 8.2 (Smooth lattice partition of unity). *For every full-rank lattice $T_0 \subset V$, every $\lambda_2 > 1$, and all sufficiently large integers s , there are $N = s^\ell$ nonzero functions*

$$h_1^\infty, \dots, h_N^\infty \in C_c^\infty(V), \quad h_i^\infty \geq 0,$$

whose periodizations form a smooth map $H^\infty : V \rightarrow \Delta^{N-1}$ and satisfy

$$\sum_{i=1}^N \sum_{\tau \in T_0} h_i^\infty(v + \tau) = 1, \quad v \in V. \quad (8.3)$$

After the same normalization carrying T_0 to $\mathbb{Z}^\ell$, every support is contained in a cube of side length less than λ_2/s .

Proof. Choose $1 < \lambda_1 < \lambda_2$ and apply Lemma 8.1. Choose a nonnegative mollifier $\rho_s \in C_c^\infty(V)$ of integral one with

$$\text{supp } \rho_s \subset B_{\varepsilon_s}^V, \quad 0 < \varepsilon_s < \frac{\lambda_2 - \lambda_1}{2s}.$$

Set $h_i^\infty = h_i * \rho_s$. The functions are nonzero, nonnegative, smooth, and compactly supported, and convolution enlarges the side length of every support cube by less than $(\lambda_2 - \lambda_1)/s$. Periodization commutes with convolution and the sums are locally finite, so

$$\begin{aligned} \sum_{i=1}^N \sum_{\tau \in T_0} h_i^\infty(v + \tau) &= \int_V \rho_s(y) \sum_{i=1}^N \sum_{\tau \in T_0} h_i(v - y + \tau) dy \\ &= \int_V \rho_s(y) dy = 1. \end{aligned}$$

The coordinate periodizations therefore define the asserted smooth simplex-valued map. $\square$

For the remainder of Sections 8 and 9, fix one of the two choices

$$\mathcal{X}(E) = C_{c, \text{PA}}(E) \quad \text{or} \quad \mathcal{X}(E) = C_c^\infty(E) \quad (8.4)$$

for every Euclidean space E , and use the same choice throughout the argument. Both choices are real vector spaces, stable under translations and finite sums. Those are the only vector-space properties used after the partition elements have been constructed; no norm, topology, or multiplicative structure is needed. The preceding two lemmas supply the additional existence statement that is essential here: arbitrarily fine nonnegative lattice partitions of unity with compact support.

8.2. Separated target slots

We next count small cubes that sit strictly inside one P_0 -fundamental parallelepiped; after shrinking, their periodic lifts form a target slot family with a uniform margin.

Lemma 8.3. *Let $P_0 \subset \mathbb{R}^\ell$ be a full-rank lattice and let F_P be a closed fundamental parallelepiped. Fix $\lambda > 0$ and put $\rho_s = \lambda/s$. For all sufficiently large s , the interior of F_P contains M_s closed cubes of side length ρ_s , with pairwise disjoint interiors, where*

$$M_s = \frac{\text{covol}(P_0)}{\lambda^\ell} s^\ell + O(s^{\ell-1}). \quad (8.5)$$

Proof. Use the grid of half-open cubes

$$\rho_s(a + [0, 1)^\ell), \quad a \in \mathbb{Z}^\ell,$$

and retain those whose closures lie in the interior of F_P . Their interiors are pairwise disjoint. Let U_s be their union. A grid cube has diameter $\sqrt{\ell} \rho_s$, so every point of F_P at distance more than $\sqrt{\ell} \rho_s$ from ∂F_P belongs to one of the retained cubes.

Because F_P is a parallelepiped, the volume of its boundary strip of width r is at most Cr for $0 < r < 1$, with a constant depending only on F_P . Indeed, after applying the inverse of the linear map defining F_P , this strip is contained in the union of 2ℓ slabs of thickness comparable to r along the faces of the unit cube. Consequently,

$$\text{covol}(P_0) - C\sqrt{\ell} \rho_s \leq |U_s| = M_s \rho_s^\ell \leq \text{covol}(P_0).$$

Divide by $\rho_s^\ell = (\lambda/s)^\ell$ to obtain (8.5). $\square$

The next lemma combines the partition-of-unity and target-slot constructions. The extra integers m, n, k are included now because Section 9 will use the same statement after the discrete rows and columns have been introduced. The fully dense case is the specialization $m = n = 1$ and $k = 0$.

Lemma 8.4 (Partition-of-unity and slot supply). *Fix one of the choices $X(V)$ in (8.4). Let $T_0, P_0 \subset V$ be full-rank lattices, let $m, n \geq 1$, and let $k \geq 0$. Put*

$$a_T = \text{covol}_V(T_0), \quad a_P = \text{covol}_V(P_0).$$

If

$$ma_T < na_P, \quad (8.6)$$

then there exist positive integers N, M , nonzero nonnegative functions $h_1, \dots, h_N \in X(V)$, bounded open slots $\widetilde{S}_1, \dots, \widetilde{S}_M \subset V$, and $\eta > 0$ such that

(i) the functions generate a T_0 -partition of unity:

$$\sum_{i=1}^N \sum_{\tau \in T_0} h_i(v + \tau) = 1, \quad v \in V; \quad (8.7)$$

(ii) for every $i \in \{1, \dots, N\}$ and $j \in \{1, \dots, M\}$ there is $y_{i,j} \in V$ with

$$(\text{supp } h_i + B_\eta^V) + y_{i,j} \subset \widetilde{S}_j; \quad (8.8)$$

(iii) $\widetilde{S}_j \cap (\widetilde{S}_{j'} + p_0) = \emptyset$ for all $j, j' \in \{1, \dots, M\}$ and $p_0 \in P_0$, unless $j = j'$ and $p_0 = 0$;

(iv)

$$nM \geq mN + k. \quad (8.9)$$

Proof. Fix a linear isomorphism $A : V \rightarrow \mathbb{R}^\ell$ with $A(T_0) = \mathbb{Z}^\ell$. In these coordinates $\alpha := \text{covol}(A(P_0)) = a_P/a_T$, so the hypothesis is $m < n\alpha$. Choose $1 < \lambda_1 < \lambda_2 < \lambda$ such that $n\alpha/\lambda^\ell > m$. For every sufficiently large s , apply Lemma 8.1 if $X(V) = C_{c,PA}(V)$ and Lemma 8.2 if $X(V) = C_c^\infty(V)$. In either case we obtain $N = s^\ell$ nonnegative partition elements in $X(V)$ whose A -images are supported in cubes of side less than λ_1/s .

Apply Lemma 8.3 to the lattice $A(P_0)$ and a closed fundamental parallelepiped for it. We obtain $M = M_s$ outer cubes of side λ/s . Inside each outer cube take the concentric open cube of side λ_2/s , and let $\widetilde{S}_j$ be its inverse image under A . The outer cubes lie strictly inside one fundamental parallelepiped and have disjoint interiors. Their shrinking therefore gives, for all $j, j' \in \{1, \dots, M\}$ and $p_0 \in P_0$ with $(j, p_0) \neq (j', 0)$,

$$\overline{\widetilde{S}_j} \cap (\overline{\widetilde{S}_{j'}} + p_0) = \emptyset.$$

This proves (iii).

For every i , the set $A(\text{supp } h_i)$ lies in a cube strictly smaller than the inner slot cube. Aligning the two centers gives a vector $y_{i,j} \in V$, and the strict difference of side lengths leaves a positive margin. Since only finitely many pairs (i, j) occur, one common $\eta > 0$ works in (ii). Item (i) is the partition identity supplied by the corresponding partition-of-unity lemma.

Finally, Lemma 8.3 gives

$$nM - mN = \left(\frac{n\alpha}{\lambda^\ell} - m \right) s^\ell + O(s^{\ell-1}).$$

The leading coefficient is positive, so the right-hand side is at least k for all sufficiently large s . This proves (iv). $\square$

8.3. Direct construction when $\overline{T + P} = \mathbb{R}^d$

We now combine the fixed function space $X(\mathbb{R}^d)$ with the density of T modulo P . No configuration field is needed in the fully dense case.

Proposition 8.5 (Fully dense construction in X). *Fix one of the choices $X(\mathbb{R}^d)$ in (8.4). Under (8.1), there are a nonnegative function $\varphi_X \in X(\mathbb{R}^d)$ and a number $\eta_X > 0$ such that φ_X is a T -partition of unity and*

$$(\text{supp } \varphi_X + B_{\eta_X}) + P \quad \text{is a packing.} \quad (8.10)$$

In particular, one may choose either

$$\varphi_{\text{pa}} \in C_{c,\text{PA}}(\mathbb{R}^d) \quad \text{or} \quad \varphi_\infty \in C_c^\infty(\mathbb{R}^d).$$

For the piecewise-affine choice, the fundamental domain in Theorem 1.1 may be chosen as a finite union of bounded half-open polyhedra.

Proof. Apply Lemma 8.4 with

$$V = \mathbb{R}^d, \quad T_0 = T, \quad P_0 = P, \quad m = n = 1, \quad k = 0,$$

and with the chosen function space $X(\mathbb{R}^d)$. The strict covolume inequality gives the hypothesis, and (8.9) gives $M \geq N$. Choose distinct slot labels $j(1), \dots, j(N)$.

Let $\rho_P : \mathbb{R}^d \rightarrow \mathbb{R}^d/P$ be the quotient map. The density assumption implies

$$\overline{\rho_P(T)} = \rho_P(\overline{T + P}) = \mathbb{R}^d/P.$$

For each i , Lemma 8.4(ii) gives a geometric placement vector $y_{i,j(i)}$. The inclusion in (8.8) is strict and therefore remains valid after a sufficiently small perturbation. Density of $\rho_P(T)$ gives $t_i \in T$ and $p_i \in P$ such that

$$(\text{supp } h_i + B_\eta) + t_i \subset \widetilde{S}_{j(i)} + p_i.$$

Define

$$\varphi_X(x) = \sum_{i=1}^N h_i(x - t_i).$$

Translation invariance and the vector-space property of $\mathcal{X}(\mathbb{R}^d)$ imply that $\varphi_{\mathcal{X}} \in \mathcal{X}(\mathbb{R}^d)$. Since $t_i \in T$ and the h_i generate a T -partition of unity,

$$\sum_{t \in T} \varphi_{\mathcal{X}}(x + t) = \sum_{i=1}^N \sum_{t \in T} h_i(x + t) = 1.$$

The slot labels $j(i)$ are distinct, and the periodic slot closures satisfy the disjointness condition in Lemma 8.4(iii). Hence (8.10) holds with $\eta_{\mathcal{X}} = \eta$.

Taking $\mathcal{X}(\mathbb{R}^d) = C_{c,PA}(\mathbb{R}^d)$ and then $\mathcal{X}(\mathbb{R}^d) = C_c^\infty(\mathbb{R}^d)$ gives the two asserted regularities. For the polyhedral conclusion, use the piecewise-affine choice. Let D be a uniform bound for the number of nonzero members of its T -partition of unity and choose $0 < \tau < 1/D$. Then $K = \{\varphi_{pa} \geq \tau\}$ is a compact finite union of polyhedra and $K + T = \mathbb{R}^d$. The first-hit construction of Subsection 5.6 selects one representative from each T -coset inside K . Its pieces are half-open polyhedra, and (8.10) supplies the positive P -packing buffer. $\square$

9. THE GENERAL PARTIALLY DENSE CASE

We now assume that

$$C = \overline{T + P}$$

is neither discrete nor all of $\mathbb{R}^d$. Thus the closed group C has both a nonzero connected factor and a nonzero discrete quotient. We give the full argument in five steps. First, a linear normal form separates those two parts. Second, the partition-of-unity and slot construction of Section 8 is applied in the connected factor. Third, the discrete quotient supplies rows and columns, and the partition-element and slot labels refine them into a larger finite table. Fourth, the extension theorem of Section 7 fills an equivariant field of soft configurations over the quotient torus. Finally, the field is unrolled to a scalar partition of unity and then replaced by piecewise-affine and smooth partitions of unity.

The connected-factor step starts from the same geometric packet–slot idea used by Grepstad, Kolountzakis, and Spyridakis [24]. In the hard version, each small source packet is assigned wholly to one separated target slot, exactly as a hard configuration assigns one address to one column. In the present partially dense setting these assignments must vary over the quotient torus, so a hard choice is too rigid. We soften it twice: the source packets are replaced by functions forming a partition of unity, and the varying slot assignments are encoded by soft configurations. The support margin inside the target slots retains the geometric separation of the hard construction.

The reason for formulating Lemma 8.4 with the auxiliary integers m, n, k is precisely this application. Here m and n will be the numbers of discrete rows and columns, while k is the dimension of the quotient torus. If the connected factor uses N compactly supported partition elements and M target slots, then the refined table has

$$\widetilde{m} = mN, \quad \widetilde{n} = nM,$$

and the inequality $nM \geq mN + k$ supplies the connectivity required to fill that k -dimensional torus.

Fix henceforth one of the function spaces $\mathcal{X}(V)$ in (8.4). All partition elements in the connected factor will belong to this space. The final replacement argument is made with the same notation and may then be run once for each of the two choices.

9.1. Closed-subgroup normal form

Let $V = C^\circ$ be the connected component of 0 in C . A connected closed subgroup of a Euclidean space is a vector subspace, so V is a linear subspace. Put

$$\ell = \dim V, \quad k = d - \ell.$$

Let

$$q : \mathbb{R}^d \longrightarrow \mathbb{R}^d / V$$

be the quotient map by the subspace V . The group $q(C)$ is a closed subgroup of the Euclidean space $\mathbb{R}^d/V$ and has trivial identity component; hence it is discrete. Since C spans $\mathbb{R}^d$, the group $q(C)$ has rank k .

Choose a linear complement W of V . The restriction $q|_W : W \rightarrow \mathbb{R}^d/V$ is a linear isomorphism. Define

$$L = (q|_W)^{-1}(q(C)).$$

Then L is a full-rank lattice in W , every $c \in C$ has a unique decomposition $c = \ell_0 + v$ with $\ell_0 \in L$ and $v \in V$, and

$$\mathbb{R}^d = W \oplus V, \quad C = L \oplus V. \quad (9.1)$$

After one common invertible linear change of variables, we may and shall assume

$$W = \mathbb{R}^k, \quad L = \mathbb{Z}^k, \quad V = \mathbb{R}^\ell.$$

We now define the projection used below without suppressing the quotient space. Let

$$q_C : C \longrightarrow C/V$$

be the quotient map of the closed group C by its connected subgroup V . The decomposition (9.1) identifies C/V with the lattice L by

$$(\ell_0 + v) + V \longmapsto \ell_0.$$

The map

$$\pi : C \longrightarrow L, \quad \pi(\ell_0 + v) = \ell_0, \quad (9.2)$$

is the composition of q_C with this identification. Set

$$T' = \pi(T), \quad P' = \pi(P), \quad T_0 = T \cap V, \quad P_0 = P \cap V.$$

For completeness, we verify the finiteness statements. Because T is a lattice in $\mathbb{R}^d$ and C is closed with $T \subset C$, the quotient C/T is a closed subset of the compact torus $\mathbb{R}^d/T$, and is therefore compact. The connected component of the identity in C/T is the image of V ; its kernel is T_0 . Thus V/T_0 is compact, so T_0 is a full-rank lattice in V . The map

$$C/T \longrightarrow L/T', \quad c + T \longmapsto \pi(c) + T',$$

is continuous and surjective. Hence L/T' is compact. Since it is also discrete, it is finite. Therefore T' has finite index in L . Repeating the same argument with P shows that P_0 is a full-rank lattice in V and that P' has finite index in L . Moreover,

$$T' + P' = L,$$

because $\pi(T + P)$ is dense in the discrete group L .

As in Section 4, enumerate the quotient groups themselves by

$$L/T' = \{R_0 = T', R_1, \dots, R_{m-1}\}, \quad L/P' = \{Q_0 = P', Q_1, \dots, Q_{n-1}\}. \quad (9.3)$$

Thus R_r are the discrete rows and Q_q are the discrete columns. Put

$$m = [L : T'], \quad n = [L : P'], \quad a_T = \text{covol}_V(T_0), \quad a_P = \text{covol}_V(P_0),$$

where covolumes with subscript V are computed using Lebesgue measure on V .

Lemma 9.1. *After applying a shear that fixes V pointwise, one may assume*

$$T = T' \times T_0.$$

In the same coordinates there is a linear map $B : W \rightarrow V$ such that

$$P = \{(p', Bp' + p_0) : p' \in P', p_0 \in P_0\}.$$

If Lebesgue measure on W is normalized by $\text{covol}_W(L) = 1$, then

$$\text{covol}(T) = ma_T, \quad \text{covol}(P) = na_P.$$

Proof. Choose a basis $t'_1, \dots, t'_k$ of the rank- k lattice T' . For each i , choose a lift $t_i \in T$ with $\pi(t_i) = t'_i$, and write

$$t_i = (t'_i, v_i), \quad v_i \in V.$$

Every $t \in T$ has $\pi(t) = \sum_i z_i t'_i$ for uniquely determined integers z_i . Therefore

$$t - \sum_i z_i t_i \in T \cap V = T_0.$$

Thus the chosen lifts together with T_0 generate all of T , and their intersection with T_0 is trivial.

Define the linear map $A : W \rightarrow V$ by $A(t'_i) = v_i$ and extend linearly from the basis $t'_1, \dots, t'_k$ of W . The shear

$$S_A : W \oplus V \longrightarrow W \oplus V, \quad S_A(w, v) = (w, v - Aw),$$

fixes every vector of V and sends each lift t_i to $(t'_i, 0)$. Hence, after applying this common change of coordinates,

$$T = T' \times T_0.$$

Now choose a basis $p'_1, \dots, p'_k$ of P' and lifts $p_i = (p'_i, w_i) \in P$. As above, every element of P differs from an integer combination of the p_i by an element of P_0 . Define the unique linear map $B : W \rightarrow V$ by $B(p'_i) = w_i$. It follows directly that every $p \in P$ has a unique representation

$$p = (p', Bp' + p_0), \quad p' \in P', \quad p_0 \in P_0,$$

which is the asserted graph form.

Finally choose lattice bases of T' , T_0 , P' , and P_0 . Relative to the decomposition $W \oplus V$, the corresponding basis matrices for T and P are block triangular. Their diagonal blocks give

$$\text{covol}(T) = \text{covol}_W(T') \text{covol}_V(T_0), \quad \text{covol}(P) = \text{covol}_W(P') \text{covol}_V(P_0).$$

Because $\text{covol}_W(L) = 1$, the finite-index formulas give $\text{covol}_W(T') = [L : T'] = m$ and $\text{covol}_W(P') = [L : P'] = n$, proving the result. $\square$

The strict covolume hypothesis is now

$$ma_T < na_P. \tag{9.4}$$

9.2. Partition of unity in the connected space

We now apply the abstract packet-slot result in the connected space V . The only regularity used at this stage is that the partition elements belong to the fixed function space $X(V) \subset C_c(V)$.

Lemma 9.2. *Assume $\ell \geq 1$ and $ma_T < na_P$. There exist positive integers N, M , nonzero nonnegative functions $h_1, \dots, h_N \in X(V)$, bounded open slot lifts $\widetilde{S}_1, \dots, \widetilde{S}_M \subset V$, and $\eta > 0$ such that*

(i) *the functions generate a T_0 -partition of unity:*

$$\sum_{i=1}^N \sum_{\tau \in T_0} h_i(v + \tau) = 1, \quad v \in V;$$

(ii) *for every $i \in \{1, \dots, N\}$ and $j \in \{1, \dots, M\}$ there is $y_{i,j} \in V$ such that*

$$(\text{supp } h_i + B_\eta^V) + y_{i,j} \subset \widetilde{S}_j;$$

(iii) *$\widetilde{S}_j \cap (\widetilde{S}_{j'} + p_0) = \emptyset$ for all $j, j' \in \{1, \dots, M\}$ and $p_0 \in P_0$, unless $j = j'$ and $p_0 = 0$;*

(iv)

$$nM \geq mN + k.$$

Proof. Apply Lemma 8.4 in the Euclidean space V to the fixed function space $X(V)$, the lattices T_0, P_0 , and the integers m, n, k defined above. Its hypothesis is exactly $ma_T < na_P$, and its four conclusions give the assertions here. $\square$

9.3. Rows, columns, and admissible addresses

We now keep the row–column notation of Sections 4 and 5. The discrete rows and columns are the cosets $R_r \in L/T'$ and $Q_q \in L/P'$ from (9.3). Refinement adds a partition-element label to a row and a slot label to a column:

$$\widetilde{R}_{r,i} = (R_r, i), \quad 0 \leq r \leq m-1, \quad 1 \leq i \leq N,$$

and

$$\widetilde{Q}_{q,j} = (Q_q, j), \quad 0 \leq q \leq n-1, \quad 1 \leq j \leq M.$$

Thus the refined table has

$$\widetilde{m} = mN \quad \text{rows and} \quad \widetilde{n} = nM$$

columns, and Lemma 9.2 gives

$$\widetilde{n} \geq \widetilde{m} + k. \quad (9.5)$$

It is useful to spell out the moving address sets explicitly. The graph form of P gives a canonical homeomorphism

$$\Theta : (W \oplus V)/P \longrightarrow (W/P') \times (V/P_0), \quad \Theta((w, v) + P) = (w + P', v - Bw + P_0). \quad (9.6)$$

Indeed, replacing (w, v) by $(w + p', v + Bp' + p_0)$ leaves the right-hand side unchanged.

For a base point $u \in W$, a refined row (R_r, i) , and a refined column (Q_q, j) , an address is a labeled quadruple

$$a = (\ell, \tau; i, j), \quad \ell \in R_r \cap Q_q, \quad \tau \in T_0.$$

The sign convention is chosen to match the unrolling formula below: the address ℓ at the base point u contributes on the affine fiber with W -coordinate $u - \ell$. Define

$$\mathcal{A}_{(r,i),(q,j)}(u) = \left\{ (\ell, \tau; i, j) : \ell \in R_r \cap Q_q, \tau \in T_0, \text{ and for some } p_0 \in P_0, \right. \\ \left. (\text{supp } h_i + \overline{B_\eta^V}) + \tau - B(u - \ell) \subset \widetilde{S}_j + p_0 \right\}. \quad (9.7)$$

After decreasing η once, if necessary, the closed-ball version of the packet inclusion follows from the strict containments in Lemma 9.2(ii). Lemma 9.2(iii) shows that the element p_0 in (9.7) is unique; denote it by $\kappa(a, u)$. Formula (9.6) explains the expression in (9.7): for a point $(u - \ell, v)$ in the translated packet, its connected coordinate modulo P is $v - B(u - \ell) + P_0$.

The definition now makes all dependence on u explicit. Since the compact set in (9.7) lies strictly inside an open slot, admissibility is open in u . If $\gamma \in L$, put

$$a + \gamma = (\ell + \gamma, \tau; i, j).$$

Then $u + \gamma - (\ell + \gamma) = u - \ell$, and hence

$$\mathcal{A}_{(R_r+\gamma,i),(Q_q+\gamma,j)}(u + \gamma) = \mathcal{A}_{(R_r,i),(Q_q,j)}(u) + \gamma. \quad (9.8)$$

Thus the moving address table is L -equivariant.

Lemma 9.3. *For every $u \in W$, every refined row $\widetilde{R}_{r,i}$, and every refined column $\widetilde{Q}_{q,j}$, the address set $\mathcal{A}_{(r,i),(q,j)}(u)$ is nonempty. More precisely, for any $\ell_0 \in R_r \cap Q_q$, the set*

$$\left\{ \tau - B(u - \ell) + P_0 : \ell \in R_r \cap Q_q, \tau \in T_0 \right\} \quad (9.9)$$

is dense in V/P_0 .

Proof. Put $H = T' \cap P'$. Since $R_r \cap Q_q = \ell_0 + H$, the set in (9.9) is a translate of the image of $T_0 + BH$ in V/P_0 . We claim that this image is dense. The elements of $T + P$ with zero discrete component are precisely

$$(0, \tau + p_0 + Bh), \quad \tau \in T_0, \quad p_0 \in P_0, \quad h \in H.$$

Because $\overline{T + P} = L \oplus V$, their connected coordinates are dense in V . Equivalently, $T_0 + BH$ is dense modulo P_0 , proving the claim and hence (9.9).

For fixed i, j , Lemma 9.2(ii) gives a nonempty open set of translation classes $y + P_0 \in V/P_0$ for which $(\text{supp } h_i + \overline{B_\eta^V}) + y$ lies in a translate of $\widetilde{S}_j$. The dense set (9.9) meets this open set. The corresponding ℓ and τ satisfy (9.7), and therefore give an address in $\mathcal{A}_{(r,i),(q,j)}(u)$. $\square$

9.4. Triangulating the moving address data

For $u \in W$, let

$$\mathcal{A}(u) = \left(\mathcal{A}_{(r,i),(q,j)}(u) \right)_{\substack{0 \leq r < m, \ 1 \leq i \leq N \\ 0 \leq q < n, \ 1 \leq j \leq M}}$$

be the refined address table. Lemma 9.3 says that every entry of this table is nonempty. An individual address a has an open admissibility set

$$U_a = \{u \in W : a \text{ is valid at } u\}.$$

The L -action relabels addresses, rows, and columns. We use the same sign convention as in Section 5:

$$(\rho_\gamma \lambda)(a) = \lambda(a - \gamma), \quad \gamma \in L.$$

Then $U_{a+\gamma} = U_a + \gamma$.

The following lemma is the moving-table counterpart of the cube construction in Section 5. Its proof is included to make clear where finiteness and continuity enter.

Lemma 9.4. *Assume that the refined table $\mathcal{A}(u)$ has $\widetilde{m}$ rows and $\widetilde{n}$ columns, that every entry is nonempty, and that address admissibility is open and L -equivariant as above. If*

$$\widetilde{n} \geq \widetilde{m} + k,$$

then there is an L -equivariant continuous simplexwise affine field Λ on W such that, for every $u \in W$, the value $\Lambda(u)$ is a soft configuration with $\widetilde{m}$ rows and $\widetilde{n}$ columns and uses only addresses belonging to $\mathcal{A}(u)$. The field uses only finitely many L -orbits of actual addresses.

Proof. The quotient W/L is a compact k -torus. At a point $\bar{u} \in W/L$, choose one address from every entry of the finite $\widetilde{m} \times \widetilde{n}$ table. This gives a finite complete table $A_{\bar{u}}$. Since all chosen admissibility sets are open, there is an open neighborhood $O_{\bar{u}}$ on which every address of $A_{\bar{u}}$ remains valid. Compactness provides finitely many such neighborhoods.

Choose a finite triangulation K of W/L sufficiently fine that the closed star of every vertex lies in one of the neighborhoods $O_{\bar{u}}$. One way to obtain this is to start from any triangulation, subdivide until its mesh is smaller than a Lebesgue number of the finite cover, and then take one barycentric subdivision. Lift K to an L -periodic triangulation $\widetilde{K}$ of W .

For one representative of every vertex orbit, choose a complete finite table that is valid on the whole closed star of that vertex, and translate these tables by L . Denote the table at a lifted vertex v by A_v . If σ is a simplex, put

$$A_\sigma = \bigcup_{v \in \sigma^{(0)}} A_v.$$

Every address in A_σ is valid at every point of σ . Moreover, if τ is a face of σ , then

$$C_{\widetilde{m}, \widetilde{n}}(A_\tau) \subset C_{\widetilde{m}, \widetilde{n}}(A_\sigma).$$

We now fill the lifted triangulation skeleton by skeleton. At each vertex choose a hard configuration from its complete table. Make these choices on one representative of every

vertex orbit and extend them L -equivariantly. Assume inductively that the field has been defined simplexwise affinely on the $(s-1)$ -skeleton. For an s -simplex σ , the already defined boundary map takes values in the finite complex

$$C_{\tilde{m}, \tilde{n}}(A_\sigma).$$

Since

$$s \leq k \leq \tilde{n} - \tilde{m},$$

Theorem 7.1 extends this boundary map over σ , after subdividing only its interior and without changing its boundary triangulation or boundary vertex values. Fill one representative of every L -orbit of s -simplices and translate the filling by the action ρ_γ .

The relative boundary condition ensures that fillings of adjacent simplices agree on their common face. Each point of W has a neighborhood meeting only finitely many simplices of the periodic triangulation, so the finite pasting lemma gives continuity. The quotient triangulation has finitely many simplices and each filling uses finitely many addresses. Therefore only finitely many address orbits occur. $\square$

9.5. Constructing the function φ

We now convert the soft-configuration field into a scalar partition of unity, in exact analogy with Subsection 5.2. Write a point of $\mathbb{R}^d = W \oplus V$ uniquely as

$$x = (u - \ell, v), \quad u \in [0, 1)^k, \quad \ell \in L, \quad v \in V.$$

An address in refined row (R_r, i) and refined column (Q_q, j) is a labeled quadruple $a = (\ell, \tau; i, j)$ belonging to $\mathcal{A}_{(r,i),(q,j)}(u)$. We write $\Lambda(u)_{r,i}(\ell, \tau, j)$ for its probability weight and let $r(\ell)$ be determined by $\ell + T' = R_{r(\ell)}$. Define

$$\varphi(u - \ell, v) = \sum_{i=1}^N \sum_{\tau \in T_0} \sum_{j=1}^M \Lambda(u)_{r(\ell), i}(\ell, \tau, j) h_i(v - \tau). \quad (9.10)$$

The sum is locally finite. Indeed, on one fundamental cube in W the field uses only finitely many L -orbits of addresses, and for v in a bounded set only finitely many T_0 -translates of the compact supports of the h_i can occur.

The equivariance relation

$$\Lambda(u + \gamma) = \rho_\gamma \Lambda(u), \quad \gamma \in L,$$

shows that (9.10) is independent of the half-open representative. Indeed, the same point can be written as $(u - \ell, v) = ((u + \gamma) - (\ell + \gamma), v)$, and equivariance gives equal weights at the addresses ℓ and $\ell + \gamma$. The formulas therefore agree on opposite faces, and φ is continuous. On a fixed fundamental cube only finitely many discrete address orbits occur; together with the compact supports of the partition elements, this shows that $\varphi \in C_c(\mathbb{R}^d)$.

We verify the T -partition-of-unity identity. Let $x = (u - \ell, v)$ and write an element of $T = T' \times T_0$ as (t', σ) . Then

$$\begin{aligned} \sum_{(t', \sigma) \in T' \times T_0} \varphi(u - \ell + t', v + \sigma) &= \sum_{i=1}^N \sum_{\ell' \in R_{r(\ell)}} \sum_{\tau \in T_0} \sum_{j=1}^M \Lambda(u)_{r(\ell), i}(\ell', \tau, j) \sum_{\sigma \in T_0} h_i(v + \sigma - \tau) \\ &= \sum_{i=1}^N \sum_{\sigma \in T_0} h_i(v + \sigma) = 1. \end{aligned}$$

Here we used $\ell' = \ell - t'$ in the first equality, the row-normalization of $\Lambda(u)$ in the second, and the connected-factor partition of unity from Lemma 9.2(i) in the last. Thus

$$\sum_{t \in T} \varphi(x + t) = 1, \quad x \in \mathbb{R}^d. \quad (9.11)$$

It remains to prove the packing identity. Let $p = (p', Bp' + p_0) \in P$ and suppose, for contradiction, that both $\varphi(x)$ and $\varphi(x - p)$ are positive. Choose positive summands in (9.10). They correspond to labeled addresses

$$a = (\ell, \tau; i, j), \quad a' = (\ell + p', \tau'; i', j'),$$

because

$$x - p = (u - (\ell + p'), v - Bp' - p_0).$$

Set $y = v - B(u - \ell)$. The definition of admissibility and positivity of the partition elements give

$$y \in \widetilde{S}_j + \kappa(a, u).$$

For the second summand,

$$(v - Bp' - p_0) - B(u - (\ell + p')) = y - p_0,$$

so

$$y - p_0 \in \widetilde{S}_{j'} + \kappa(a', u), \quad \text{or equivalently} \quad y \in \widetilde{S}_{j'} + \kappa(a', u) + p_0.$$

The disjointness of the closed periodic slot translates in Lemma 9.2(iii) therefore forces

$$j = j', \quad \kappa(a, u) = \kappa(a', u) + p_0. \quad (9.12)$$

The discrete addresses ℓ and $\ell + p'$ lie in the same class modulo P' , and (9.12) says that their slot labels also agree. Hence a and a' lie in the same refined column. If they are distinct labeled addresses, this contradicts column exclusion. If they are the same labeled address, then $p' = 0$ and $\kappa(a, u) = \kappa(a', u)$; (9.12) then gives $p_0 = 0$. In either case a nonzero p is impossible. Therefore

$$\varphi(x)\varphi(x - p) = 0, \quad p \in P \setminus \{0\}. \quad (9.13)$$

The function in (9.10) is continuous, but it need not be piecewise affine: even when Λ and the partition elements are piecewise affine separately, their products in the base and fiber variables are generally piecewise bilinear. Choosing the partition elements smooth does not make the result smooth either, because the configuration field is only simplexwise affine in the base variable. The following replacement lemma produces both regularities after the exact partition-of-unity and separation identities have been established.

9.6. Piecewise-affine and smooth replacement

The intermediate product construction is only continuous, so we now replace it by partitions of unity with the two regularities needed in the applications.

Lemma 9.5 (Regularity replacement in $\mathcal{X}$). *Fix one of the choices $\mathcal{X}(\mathbb{R}^d)$ in (8.4). Let $T, P \subset \mathbb{R}^d$ be full-rank lattices. Suppose that $\varphi \in C_c(\mathbb{R}^d)$ is a nonnegative T -partition of unity and satisfies*

$$\varphi(x)\varphi(x - p) = 0, \quad x \in \mathbb{R}^d, \quad p \in P \setminus \{0\}. \quad (9.14)$$

Then there exist a nonnegative function $\psi_{\mathcal{X}} \in \mathcal{X}(\mathbb{R}^d)$ and a number $\eta_{\mathcal{X}} > 0$ such that $\psi_{\mathcal{X}}$ is a T -partition of unity and

$$(\text{supp } \psi_{\mathcal{X}} + B_{\eta_{\mathcal{X}}}) + P \quad \text{is a packing.} \quad (9.15)$$

Proof. Compactness of $\text{supp } \varphi$ gives a finite number

$$D = \sup_{x \in \mathbb{R}^d} \#\{t \in T : \varphi(x + t) > 0\}.$$

Choose $0 < \tau < 1/D$ and set

$$K = \{x : \varphi(x) \geq \tau\}.$$

Because φ is a T -partition of unity, at least one member of the partition is at least $1/D$ at every point. Hence

$$K + T = \mathbb{R}^d. \quad (9.16)$$

By (9.14), the compact sets K and $K + p$ are disjoint for every $p \in P \setminus \{0\}$. Since P is discrete and K is bounded,

$$\delta = \inf_{p \in P \setminus \{0\}} \text{dist}(K, K + p) > 0. \quad (9.17)$$

Choose $0 < r < \delta/3$ and a T -periodic locally finite triangulation of $\mathbb{R}^d$ whose closed vertex stars have diameter less than r . There are only finitely many vertex orbits modulo T ; choose representatives $v_1, \dots, v_s$. By (9.16), for every j there is $t_j \in T$ such that

$$v_j + t_j \in K.$$

Assign the value 1 to the vertex $v_j + t_j$ and the value 0 to every other vertex in the orbit $v_j + T$. Extend these values affinely over every simplex. The resulting function ψ_{pa} is nonnegative and continuous. Only the closed stars of the finitely many selected vertices can meet its support; hence

$$\text{supp } \psi_{\text{pa}} \subset K + B_r,$$

so it is compactly supported and piecewise affine.

Let $\sigma = [w_0, \dots, w_q]$ be a simplex and write $x = \sum_i \alpha_i w_i$ in barycentric coordinates. Periodicity of the triangulation gives the same barycentric coordinates for $x + t$ in $\sigma + t$. Therefore

$$\sum_{t \in T} \psi_{\text{pa}}(x + t) = \sum_i \alpha_i \sum_{t \in T} \psi_{\text{pa}}(w_i + t) = \sum_i \alpha_i = 1,$$

because exactly one vertex in every T -orbit was assigned the value 1. Thus ψ_{pa} is a T -partition of unity. Moreover,

$$\text{dist}(\text{supp } \psi_{\text{pa}}, \text{supp } \psi_{\text{pa}} + p) \geq \delta - 2r > 0, \quad p \in P \setminus \{0\}.$$

Choose $0 < \eta_{\text{pa}} < (\delta - 2r)/2$.

If $\mathcal{X}(\mathbb{R}^d) = C_{c,\text{PA}}(\mathbb{R}^d)$, set $\psi_{\mathcal{X}} = \psi_{\text{pa}}$ and $\eta_{\mathcal{X}} = \eta_{\text{pa}}$. If $\mathcal{X}(\mathbb{R}^d) = C_c^\infty(\mathbb{R}^d)$, choose a nonnegative mollifier $\rho \in C_c^\infty(B_\varepsilon)$ of integral one, where $0 < \varepsilon < \eta_{\text{pa}}/3$, and set

$$\psi_{\mathcal{X}} = \psi_{\text{pa}} * \rho.$$

Then $\psi_{\mathcal{X}}$ is nonnegative, smooth, and compactly supported. Since periodization commutes with convolution,

$$\begin{aligned} \sum_{t \in T} \psi_{\mathcal{X}}(x + t) &= \int_{\mathbb{R}^d} \rho(y) \sum_{t \in T} \psi_{\text{pa}}(x - y + t) dy \\ &= 1. \end{aligned}$$

Moreover,

$$\text{supp } \psi_{\mathcal{X}} \subset \text{supp } \psi_{\text{pa}} + B_\varepsilon.$$

For any $0 < \eta_{\mathcal{X}} < \eta_{\text{pa}} - \varepsilon$, we obtain the required P -packing buffer. $\square$

Apply Lemma 9.5 twice to the continuous function constructed in (9.10), using (9.11) and (9.13): once with $\mathcal{X}(\mathbb{R}^d) = C_{c,\text{PA}}(\mathbb{R}^d)$ and once with $\mathcal{X}(\mathbb{R}^d) = C_c^\infty(\mathbb{R}^d)$. Denote the two outputs by ψ_{pa} and ψ_∞ , and their buffers by η_{pa} and η_∞ . The polyhedral selector uses ψ_{pa} .

Let

$$D_\psi = \sup_x \# \{t \in T : \psi_{\text{pa}}(x + t) > 0\}$$

and choose $0 < \tau_\psi < 1/D_\psi$. Then

$$K_\psi = \{x : \psi_{\text{pa}}(x) \geq \tau_\psi\}$$

is a compact finite union of polyhedra and $K_\psi + T = \mathbb{R}^d$. Choose a half-open T -fundamental parallelepiped F . Only finitely many translates $F + t_1, \dots, F + t_N$ meet K_ψ . Put

$$D_j = \{x \in F : x + t_j \in K_\psi\}, \quad E_1 = D_1, \quad E_j = D_j \setminus \bigcup_{i < j} D_i,$$

and define

$$\Omega = \bigcup_{j=1}^N (E_j + t_j). \quad (9.18)$$

The sets E_j form a disjoint partition of F , so Ω contains exactly one representative of every T -coset and $\Omega + T$ is an exact tiling. Since K_ψ is polyhedral, the E_j and hence Ω are finite unions of bounded half-open polyhedra. Moreover, $\overline{\Omega} \subset K_\psi \subset \text{supp } \psi_{\text{pa}}$, so any sufficiently small $0 < \varepsilon < \eta_{\text{pa}}$ makes $\Omega + B_\varepsilon$ pack with P by (9.15), applied with $\mathcal{X}(\mathbb{R}^d) = C_{c,\text{PA}}(\mathbb{R}^d)$.

Corollary 9.6. *Whenever the condition (TeP) holds, there exist both a nonnegative continuous compactly supported piecewise-affine T -partition of unity φ_{pa} and a nonnegative compactly supported smooth T -partition of unity φ_∞ . Moreover, there are $\eta_{\text{pa}}, \eta_\infty > 0$ such that*

$$(\text{supp } \varphi_\star + B_{\eta_\star}) + P \quad \text{is a packing}, \quad \star \in \{\text{pa}, \infty\}.$$

Proof. If $T + P$ is discrete, Section 5 constructs a piecewise-affine T -partition of unity directly; apply Lemma 9.5 once for each choice in (8.4). If $T + P$ is dense in $\mathbb{R}^d$, apply Proposition 8.5 once for each choice. In the remaining non-discrete case, the conclusion is the pair of applications of Lemma 9.5 made above. $\square$

This completes the intermediate non-discrete construction.

10. NECESSITY IN THE TILING- ε -PACKING THEOREM

The necessity argument has two parts: the strict covolume obstruction and, when $T + P$ is discrete, the finite-index obstruction.

10.1. The strict covolume obstruction

If a bounded T -fundamental domain Ω satisfies that $\Omega + B_\varepsilon$ packs with P , then volume comparison gives

$$\text{covol}(T) = |\Omega| < |\Omega + B_\varepsilon| \leq \text{covol}(P).$$

This is the strict covolume obstruction stated after Theorem 1.1.

10.2. The discrete index obstruction

Assume that $H = T + P$ is a lattice. After a common invertible linear change of variables, we may suppose $H = \mathbb{Z}^d$. For this proof only, enumerate the cosets as

$$\mathbb{Z}^d / T = \{R_0 = T, R_1, \dots, R_{m-1}\}, \quad \mathbb{Z}^d / P = \{Q_0 = P, Q_1, \dots, Q_{n-1}\},$$

where

$$m = [\mathbb{Z}^d : T], \quad n = [\mathbb{Z}^d : P].$$

No configuration-space terminology is needed in the necessity argument.

Theorem 10.1. *If $n \leq m + d - 1$, then there is no Borel T -fundamental domain Ω and no $\varepsilon > 0$ such that $\Omega + B_\varepsilon$ packs with P .*

The proof uses the classical covering-dimension theorem of Lebesgue: a finite closed cover of a d -cube by sets of sufficiently small mesh must have multiplicity at least $d + 1$. We use the following standard form; see [31, Chapter IV].

Theorem 10.2 (Lebesgue [31]). *Let $Q \subset \mathbb{R}^d$ be a cube. If a finite closed cover of Q has mesh strictly smaller than the side length of Q , then some point of Q belongs to at least $d + 1$ members of the cover.*

Proof of Theorem 10.1. Assume, to obtain a contradiction, that such Ω and ε exist. We may assume that Ω is bounded: otherwise one can choose infinitely many points of Ω separated by more than 2ε , so $\Omega + B_\varepsilon$ contains infinitely many disjoint balls of radius $\varepsilon/2$ and has infinite measure, which is impossible for a P -packing. Choose representatives

$$u_i \in R_i, \quad i = 0, \dots, m-1, \quad u_0 = 0.$$

Because $\Omega + T$ is an exact tiling of $\mathbb{R}^d$, the closed translates

$$\{\overline{\Omega} + t : t \in T\}$$

cover $\mathbb{R}^d$. Their diameters are uniformly bounded by $\text{diam}(\overline{\Omega})$. Every bounded cube meets only finitely many of them: if a cube Q meets $\overline{\Omega} + t$, then $t \in Q - \overline{\Omega}$, which is bounded, and the lattice T has only finitely many points in a bounded set.

Choose a cube Q whose side length is strictly larger than $\text{diam}(\overline{\Omega})$, and cover it by the finitely many closed sets $Q \cap (\overline{\Omega} + t)$ that meet it. The cover has mesh smaller than the side length of Q . Theorem 10.2 gives distinct $t_0, \dots, t_d \in T$ and $x_0 \in Q$ such that

$$x_0 \in \bigcap_{j=0}^d (\overline{\Omega} + t_j).$$

For each $i = 1, \dots, m-1$, the family

$$\{\overline{\Omega} + t + u_i : t \in T\}$$

is a translate of the preceding closed cover. Choose $\widetilde{t}_i \in T$ such that

$$x_0 \in \overline{\Omega} + \widetilde{t}_i + u_i.$$

The elements

$$t_0, \dots, t_d, \widetilde{t}_1 + u_1, \dots, \widetilde{t}_{m-1} + u_{m-1}$$

are $m + d$ distinct elements of $\mathbb{Z}^d$: the first $d + 1$ lie in the zero class modulo T and are distinct, while the remaining elements lie in the pairwise distinct nonzero cosets R_i .

There are only $n \leq m + d - 1$ classes modulo P . Hence two of the displayed elements, say $z \neq z'$, belong to the same coset Q_j modulo P . Therefore

$$p_0 = z - z' \in P \setminus \{0\}.$$

By construction,

$$x_0 \in (\overline{\Omega} + z) \cap (\overline{\Omega} + z').$$

Subtracting z' gives a point in $(\overline{\Omega} + p_0) \cap \overline{\Omega}$. Since $\overline{\Omega} \subset \Omega + B_\varepsilon$, this contradicts the assumed P -packing. $\square$

This completes Part A.

Part B Existence of Schwartz-Class and C_c^∞ Gabor Windows

11. GABOR SYSTEMS, DUALITY, AND ZAK-TRANSFORM PRELIMINARIES

This section collects the general Gabor-theoretic and Zak-transform facts used throughout Part B. The definitions of time shifts, frequency shifts, Gabor systems, frames, tight frames, and Parseval frames are those introduced in Subsection 1.2. Tools used later, such as the symplectic normal form or metaplectic reduction, are introduced only after the simplest case of diagonal lattices has been fully resolved.

The frame operator of a Gabor frame $\mathcal{G}(g, \Lambda)$ is

$$S_{g,\Lambda}f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)g.$$

The series converges unconditionally in $L^2(\mathbb{R}^d)$. The frame is Parseval precisely when $S_{g,\Lambda} = I$.

Let $S = S_{g,\Lambda}$. Since S is positive and invertible, its positive square root $S^{1/2}$ and inverse square root $S^{-1/2}$ are well defined. The canonical tight window

$$\widetilde{g} = S^{-1/2}g$$

generates a Parseval Gabor frame on the same lattice. If $g \in \mathcal{S}(\mathbb{R}^d)$, then $S^{-1/2}g \in \mathcal{S}(\mathbb{R}^d)$; see [32, 39] and [25, Chapters 12–13]. Conversely, every Parseval frame is tight and every tight frame is a frame, while a tight frame with bound $A > 0$ becomes Parseval after replacing g by $A^{-1/2}g$. Consequently, on a fixed lattice, a Schwartz frame window exists if and only if a Schwartz Parseval frame window exists.

For a finite family $\mathbf{g} = (g_1, \dots, g_q)$, the multiwindow frame operator is obtained by summing the corresponding operators over $j = 1, \dots, q$. We refer to [25, Chapters 5–7] for the standard frame-operator theory, canonical dual and tight windows, and unconditional Gabor expansions.

Recall from Subsection 1.2 that the adjoint lattice of a full-rank phase-space lattice Λ is

$$\Lambda^\circ = \{z \in \mathbb{R}^{2d} : \sigma(z, \lambda) \in \mathbb{Z} \text{ for all } \lambda \in \Lambda\}.$$

It satisfies $\text{covol}(\Lambda^\circ) = \text{covol}(\Lambda)^{-1}$. For a separable lattice $\Lambda = \Gamma \times \Phi$, one has $\Lambda^\circ = \Phi^* \times \Gamma^*$. We use the following normalized form of the duality principle; see [32, 33, 42] and [25, Chapter 7].

Theorem 11.1 (Janssen–Ron–Shen [33, 42]). *The system $\mathcal{G}(g, \Lambda)$ is a Parseval frame if and only if $\mathcal{G}(\text{covol}(\Lambda)^{-1/2}g, \Lambda^\circ)$ is an orthonormal system.*

The density theorem gives the basic restriction: if $\mathcal{G}(g, \Lambda)$ is a frame, then $\text{covol}(\Lambda) \leq 1$; see [25, Chapters 7–8]. At critical density, $\text{covol}(\Lambda) = 1$, the Balian–Low phenomenon excludes well-localized windows. In particular, a frame window cannot belong to $S_0(\mathbb{R}^d)$, and hence cannot belong to $\mathcal{S}(\mathbb{R}^d)$; see [3, 4, 7, 18, 26, 28, 38].

11.1. The Zak transform and continuous-Zak windows

For general Zak-transform background and its role in Gabor analysis, see [25, Chapter 8]. We use the convention

$$Zf(x, \omega) = \sum_{k \in \mathbb{Z}^d} f(x - k) e^{2\pi i \langle k, \omega \rangle}, \quad x, \omega \in \mathbb{R}^d. \quad (11.1)$$

Proposition 11.2. *Let $f \in L^2(\mathbb{R}^d)$.*

(i) *The Zak transform extends to a unitary map*

$$Z : L^2(\mathbb{R}^d) \longrightarrow L^2([0, 1)^{2d}).$$

(ii) *For $n, m \in \mathbb{Z}^d$,*

$$Zf(x + n, \omega) = e^{2\pi i \langle n, \omega \rangle} Zf(x, \omega), \quad (11.2)$$

$$Zf(x, \omega + m) = Zf(x, \omega). \quad (11.3)$$

(iii) *For $n \in \mathbb{Z}^d$ and $y \in \mathbb{R}^d$,*

$$Z(M_n T_y f)(x, \omega) = e^{2\pi i \langle n, x \rangle} Zf(x - y, \omega). \quad (11.4)$$

(iv) *If $f \in \mathcal{S}(\mathbb{R}^d)$, then Zf is smooth on $\mathbb{R}^{2d}$, and all partial derivatives may be obtained by differentiating the defining series term by term.*

(v) *One has*

$$S_0(\mathbb{R}^d) \subset W_0(\mathbb{R}^d) \subset W(C, \ell^1)(\mathbb{R}^d) \subset C_Z(\mathbb{R}^d).$$

More precisely, if $f \in W(C, \ell^1)(\mathbb{R}^d)$, then the Zak series converges absolutely and uniformly on $[0, 1]^d \times [0, 1]^d$, and its quasiperiodic extension is continuous on $\mathbb{R}^{2d}$.

(vi) *If $f \in C_Z(\mathbb{R}^d)$, then the identities (11.2)–(11.3) hold everywhere for the chosen continuous representative.*

(vii) *With the Fourier-transform convention $\mathcal{F}f(\omega) = \widehat{f}(\omega) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x, \omega \rangle} dx$, one has*

$$Z(\widehat{f})(x, \omega) = e^{2\pi i \langle x, \omega \rangle} Zf(-\omega, x) \quad (11.5)$$

in the L^2 sense, and hence pointwise whenever the two sides have continuous representatives. In particular, the Fourier transform maps $C_Z(\mathbb{R}^d)$ onto itself.

These are standard Zak-transform and Wiener-amalgam facts. Proofs of the unitarity, quasiperiodicity, covariance, smoothness, Fourier covariance, and uniform-convergence statements may be found in [25, Chapter 8 and Chapters 11–12]. The notation W_0 for the Fourier-invariant Wiener class is the one used in [13].

11.2. Scalar common zeros and the rectangular Zak gap

A continuous scalar function

$$s : \mathbb{R}^q \times \mathbb{R}^q \longrightarrow \mathbb{C}$$

is called *Zak-quasiperiodic* if

$$s(u + n, v + m) = e^{2\pi i \langle n, v \rangle} s(u, v), \quad n, m \in \mathbb{Z}^q. \quad (11.6)$$

We use the following common-zero theorem of de Dios Pont, Liehr, and Taylor.

Theorem 11.3 (de Dios Pont–Liehr–Taylor [13]). *Let $s : \mathbb{R}^q \times \mathbb{R}^q \rightarrow \mathbb{C}^M$ be continuous and componentwise Zak-quasiperiodic. If $M \leq q$, then s has a zero. If $M > q$, there are smooth such maps with no zero.*

Appendix A gives an independent first-principles proof of the implication $M \leq q$. We now use both parts of Theorem 11.3 to derive the rectangular rank gap needed below. For completeness, we first give the short explicit construction for the second assertion. Put

$$\rho(t) = \begin{cases} 0, & t \leq 0, \\ e^{-1/t}, & t > 0, \end{cases} \quad \chi(t) = \frac{\rho(t)}{\rho(t) + \rho(1-t)}.$$

Then $\chi \in C^\infty(\mathbb{R})$, $\chi = 0$ on $(-\infty, 0]$, $\chi = 1$ on $[1, \infty)$, every derivative of χ vanishes at 0 and 1, and $\chi(1-t) = 1 - \chi(t)$. On $0 \leq t \leq 1$ set $h_*(t, v) = 1 - \chi(t) + \chi(t)e^{2\pi i v}$. If $u = n + t$ with $n \in \mathbb{Z}$ and $0 \leq t < 1$, define

$$h(u, v) = e^{2\pi i n v} (1 - \chi(t) + \chi(t)e^{2\pi i v}).$$

Flatness at the endpoints makes this quasiperiodic extension smooth, and its only zero modulo $\mathbb{Z}^2$ is $(1/2, 1/2)$. For $\beta \in \mathbb{R}$, put $h_\beta(u, v) = e^{2\pi i \beta u} h(u, v - \beta)$; its zero is $(1/2, 1/2 + \beta)$. Taking $\beta_r = r/(q+1)$ and

$$s_r(u, v) = \prod_{j=1}^q h_{\beta_r}(u_j, v_j), \quad r = 0, \dots, q,$$

gives $q+1$ smooth Zak-quasiperiodic components without a common zero, since each coordinate can annihilate at most one component. Appending zero components yields examples for every $M > q+1$.

For the corresponding continuous example one may use the identity cutoff $\chi(t) = t$, which gives the transparent formula $h_*(t, v) = 1 - t + te^{2\pi i v}$ and the same zero. Its quasiperiodic extension has a derivative jump at the integers, however, so the identity cannot be used for the smooth assertion of the theorem; the flat cutoff above is used throughout the smooth construction.

Theorem 11.4. *Let*

$$S : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow M_{N,R}(\mathbb{C})$$

be continuous and satisfy

$$S(u+n, v+m) = e^{2\pi i \langle n, v \rangle} S(u, v), \quad n, m \in \mathbb{Z}^d. \quad (11.7)$$

If $S(u, v)$ has full column rank R for every (u, v) , then

$$N \geq R + d. \quad (11.8)$$

Conversely, whenever $N \geq R + d$, there exists a smooth map satisfying (11.7) and having full column rank everywhere. Thus the bound is sharp.

Proof. If $R = 1$, the N entries of the single column are continuous Zak-quasiperiodic functions in dimension d with no common zero. The first part of Theorem 11.3 gives $N > d$, hence $N \geq d + 1 = R + d$.

Assume $R \geq 2$. Set $q = R - 1$ and apply the second part of Theorem 11.3 with $M = R$ in dimension q . It provides smooth Zak-quasiperiodic functions $\rho_1, \dots, \rho_R$ with no common zero. Thus the vector

$$\rho(x, y) := (\rho_1(x, y), \dots, \rho_R(x, y))^T \neq 0, \quad (x, y) \in \mathbb{R}^q \times \mathbb{R}^q. \quad (11.9)$$

For each row index $\ell = 1, \dots, N$, define a scalar function in $d + q$ Zak dimensions by

$$F_\ell((u, x), (v, y)) = \sum_{k=1}^R S_{\ell k}(u, v) \rho_k(x, y). \quad (11.10)$$

In vector form,

$$(F_1, \dots, F_N)^T = S(u, v) \rho(x, y). \quad (11.11)$$

Let $n, m \in \mathbb{Z}^d$ and $p, r \in \mathbb{Z}^q$. The quasiperiodicity of S and ρ gives

$$\begin{aligned} F_\ell((u+n, x+p), (v+m, y+r)) &= e^{2\pi i \langle n, v \rangle} e^{2\pi i \langle p, y \rangle} F_\ell((u, x), (v, y)) \\ &= e^{2\pi i \langle (n, p), (v, y) \rangle} F_\ell((u, x), (v, y)). \end{aligned}$$

Thus $F_1, \dots, F_N$ are continuous Zak-quasiperiodic functions in $d + R - 1$ dimensions.

They have no common zero. Indeed, if they all vanished at some point, then (11.11) would give

$$S(u, v) \rho(x, y) = 0.$$

Full column rank means that the kernel of $S(u, v)$ is zero, so this would force $\rho(x, y) = 0$, contrary to (11.9).

The first half of Theorem 11.3, applied in dimension $q' = d + q = d + R - 1$, therefore forces $N > d + R - 1$. All quantities are integers, hence $N \geq R + d$, which is (11.8).

For the converse, take smooth Zak-quasiperiodic functions $s_0, \dots, s_d$ with no common zero from the explicit construction after Theorem 11.3. First suppose $N = R + d$. Index the rows by $0, \dots, R + d - 1$ and the columns by $0, \dots, R - 1$, and define

$$(S_0(u, v))_{\ell k} = \begin{cases} s_{\ell-k}(u, v), & 0 \leq \ell - k \leq d, \\ 0, & \text{otherwise.} \end{cases}$$

Every entry has the required quasiperiodicity. To check the rank, fix (u, v) and suppose $S_0(u, v)c = 0$, where $c = (c_0, \dots, c_{R-1})^T$. Form the two polynomials

$$a(z) = \sum_{j=0}^d s_j(u, v)z^j, \quad c(z) = \sum_{k=0}^{R-1} c_k z^k.$$

The entries of $S_0(u, v)c$ are exactly the coefficients of the product $a(z)c(z)$. Since the s_j have no common zero, a is not the zero polynomial. The ring $\mathbb{C}[z]$ has no zero divisors, so $c(z) = 0$ and hence $c = 0$. Thus S_0 has full column rank everywhere. If $N > R + d$, append $N - R - d$ zero rows. This completes the sharp converse. Notice that when (11.8) fails, the first part of the theorem rules out an everywhere full-rank map; the existence assertion is precisely for the range in which the inequality holds. $\square$

12. THE DIAGONAL SYMPLECTICALLY RATIONAL MODEL: EXISTENCE AND NECESSITY

With the general Gabor and Zak-transform tools in place, we now prove the one-window result for the diagonal lattice

$$\Lambda_D = D\mathbb{Z}^d \times \mathbb{Z}^d, \quad D = \text{diag}\left(\frac{b_1}{a_1}, \dots, \frac{b_d}{a_d}\right), \quad (a_i, b_i) = 1. \quad (12.1)$$

All a_i, b_i are positive integers. Put $N = \prod_{i=1}^d a_i$, $R = \prod_{i=1}^d b_i$. For the sufficiency direction we work in the nontrivial density range

$$\text{covol}(\Lambda_D) = \det D = \frac{R}{N} < 1, \quad \text{density}(\Lambda_D) = \frac{N}{R} > 1. \quad (12.2)$$

Lemma 12.1. *For the rational lattice Λ_D in (12.1),*

$$v(\Lambda_D) = N, \quad v(\Lambda_D^\circ) = R, \quad \text{covol}(\Lambda_D) = \frac{R}{N}. \quad (12.3)$$

Consequently, the condition (SG) for Λ_D is equivalent to

$$N \geq R + d. \quad (12.4)$$

Proof. Write a point of $\mathbb{R}^{2d}$ as (x, ξ) . By the definition of the adjoint lattice, $(x, \xi) \in \Lambda_D^\circ$ if and only if

$$\langle n, x \rangle \in \mathbb{Z} \quad \text{for every } n \in \mathbb{Z}^d, \quad \langle \xi, Dk \rangle \in \mathbb{Z} \quad \text{for every } k \in \mathbb{Z}^d.$$

Hence

$$\Lambda_D^\circ = \mathbb{Z}^d \times D^{-1}\mathbb{Z}^d. \quad (12.5)$$

Let $A = \text{diag}(a_1, \dots, a_d)$ and $B = \text{diag}(b_1, \dots, b_d)$. Coprimality gives, in each coordinate,

$$\frac{b_i}{a_i}\mathbb{Z} \cap \mathbb{Z} = b_i\mathbb{Z}, \quad \mathbb{Z} \cap \frac{a_i}{b_i}\mathbb{Z} = a_i\mathbb{Z}.$$

Therefore

$$\Lambda_{D,\text{int}} = \Lambda_D \cap \Lambda_D^\circ = B\mathbb{Z}^d \times A\mathbb{Z}^d. \quad (12.6)$$

The index factors over the two coordinates:

$$[\Lambda_D : \Lambda_{D,\text{int}}] = [D\mathbb{Z}^d : B\mathbb{Z}^d][\mathbb{Z}^d : A\mathbb{Z}^d] = N^2,$$

so $v(\Lambda_D) = N$. The same intersection is the integral subgroup of Λ_D° , and

$$[\Lambda_D^\circ : \Lambda_{D,\text{int}}] = [\mathbb{Z}^d : B\mathbb{Z}^d][D^{-1}\mathbb{Z}^d : A\mathbb{Z}^d] = R^2.$$

Thus $v(\Lambda_D^\circ) = R$. Finally, $\text{covol}(\Lambda_D) = \det D = R/N$, and substituting these identities into (SG) yields $N \geq R + d$. $\square$

12.1. Compactly supported Parseval windows in the diagonal model (proof of Theorem 1.7(b) in diagonal form)

Assume that the condition (SG) holds for Λ_D . By Lemma 12.1, the condition (TeP) holds for the pair $(D\mathbb{Z}^d, \mathbb{Z}^d)$. The following theorem is the diagonal existence result.

Theorem 12.2. *Let D satisfy (12.1). If $N \geq R + d$, then there exists a nonnegative $g \in C_c^\infty(\mathbb{R}^d)$ such that $\mathcal{G}(g, D\mathbb{Z}^d \times \mathbb{Z}^d)$ is a Parseval frame. With $\Gamma = D\mathbb{Z}^d$ and $\Phi = \mathbb{Z}^d$, the window can be chosen to satisfy*

$$\text{supp } g \cap (\text{supp } g + \xi^*) = \emptyset, \quad \xi^* \in \Phi^* \setminus \{0\}, \quad (12.7)$$

$$\sum_{\gamma \in \Gamma} |g(x + \gamma)|^2 = \text{covol}(\Phi), \quad x \in \mathbb{R}^d. \quad (12.8)$$

Equivalently, $\text{covol}(\Phi)^{-1}|g|^2$ is a smooth Γ -partition of unity. Alternatively, there is a nonnegative $g_{\text{pa}} \in C_c(\mathbb{R}^d)$ for which $\text{covol}(\Phi)^{-1}g_{\text{pa}}^2$ is a continuous piecewise-affine Γ -partition of unity and (12.7)–(12.8) hold with g_{pa} in place of g .

Proof. Corollary 9.6 supplies a nonnegative compactly supported piecewise-affine Γ -partition of unity φ satisfying

$$\sum_{\gamma \in \Gamma} \varphi(x + \gamma) = 1, \quad (12.9)$$

$$(\text{supp } \varphi + B_\eta) + \Phi^* \text{ is a packing} \quad (12.10)$$

for some $\eta > 0$. The unsmoothed alternative is immediate: put

$$g_{\text{pa}}(x) = \left(\text{covol}(\Phi) \varphi(x) \right)^{1/2}. \quad (12.11)$$

Then $\text{covol}(\Phi)^{-1}g_{\text{pa}}^2 = \varphi$ is the piecewise-affine Γ -partition of unity, $\text{supp } g_{\text{pa}} = \text{supp } \varphi$, and (12.7)–(12.8) hold. The standard adjoint-lattice argument below therefore shows that $\mathcal{G}(g_{\text{pa}}, \Gamma \times \Phi)$ is Parseval.

To obtain a smooth window, we mollify a positive part of the same piecewise-affine partition of unity while preserving its positive support separation, and then normalize its energy.

Because φ has compact support, there is a finite integer D_0 such that at most D_0 terms in (12.9) are nonzero at any point. Consequently, a single translate of φ must contribute at least $1/D_0$ to the constant sum. Choose

$$0 < a < b < c < \frac{1}{D_0}$$

and set

$$K_b = \{\varphi \geq b\}, \quad V_a = \{\varphi > a\}.$$

Then K_b is compact and $K_b \subseteq V_a$, where $\subseteq$ denotes compact containment, and $\{\varphi \geq c\} + \Gamma = \mathbb{R}^d$. For a real number r , write $r_+ = \max\{r, 0\}$, and set

$$h = (\varphi - b)_+.$$

Choose a standard nonnegative mollifier $\rho \in C_c^\infty(B_1)$ with $\int \rho = 1$, put $\rho_\delta(x) = \delta^{-d}\rho(x/\delta)$, and define

$$\psi = h * \rho_\delta.$$

For sufficiently small $\delta > 0$,

$$\psi \in C_c^\infty(\mathbb{R}^d), \quad \text{supp } \psi \subset V_a,$$

and $\psi > 0$ on $\{\varphi \geq c\}$. Hence $\{\psi > 0\} + \Gamma = \mathbb{R}^d$. The support remains Φ^* -separated, because $\text{supp } \psi \subset V_a$ and (12.10) excludes positive values at points differing by a nonzero shift of Φ^* .

Define

$$D_\psi(x) = \sum_{\gamma \in \Gamma} \psi(x + \gamma)^2.$$

This function is smooth, strictly positive, and Γ -periodic. Put

$$g(x) = \sqrt{\text{covol}(\Phi)} \frac{\psi(x)}{\sqrt{D_\psi(x)}}.$$

Then $g \in C_c^\infty(\mathbb{R}^d)$, $g \geq 0$, and (12.7)–(12.8) hold.

It remains to verify the Parseval property. Let $h_0 = \text{covol}(\Gamma \times \Phi)^{-1/2} g$. Then

$$\sum_{\gamma \in \Gamma} |h_0(x + \gamma)|^2 = \frac{1}{\text{covol}(\Gamma)},$$

and the Φ^* -support separation is unchanged. We show that $\mathcal{G}(h_0, \Phi^* \times \Gamma^*)$ is orthonormal. Distinct translations by Φ^* have disjoint supports. For a fixed translation and $\gamma^*, \gamma'^* \in \Gamma^*$, periodization over a fundamental domain F_Γ gives

$$\begin{aligned} \langle h_0, M_{\gamma'^* - \gamma^*} h_0 \rangle &= \int_{F_\Gamma} \sum_{\gamma \in \Gamma} |h_0(x + \gamma)|^2 e^{2\pi i \langle \gamma'^* - \gamma^*, x + \gamma \rangle} dx \\ &= \frac{1}{\text{covol}(\Gamma)} \int_{F_\Gamma} e^{2\pi i \langle \gamma'^* - \gamma^*, x \rangle} dx = \delta_{\gamma^*, \gamma'^*}. \end{aligned}$$

Theorem 11.1 now gives that $\mathcal{G}(g, \Gamma \times \Phi) = \mathcal{G}(g, D\mathbb{Z}^d \times \mathbb{Z}^d)$ is Parseval. $\square$

12.2. Necessity of the (SG) gap in the diagonal model (proof of Theorem 1.7(d))

We now assume only that the Zak transform of the window is continuous. The next theorem proves directly that a one-window frame on the diagonal lattice must satisfy the algebraic gap.

Theorem 12.3. *Let D be as in (12.1), and let $g \in C_Z(\mathbb{R}^d)$. If $\mathcal{G}(g, D\mathbb{Z}^d \times \mathbb{Z}^d)$ is a frame, then $N \geq R + d$, that is, the condition (SG) is satisfied.*

Proof. Write

$$\begin{aligned} D &= BA^{-1}, \quad A = \text{diag}(a_1, \dots, a_d), \quad B = \text{diag}(b_1, \dots, b_d), \\ N &= \det A = \prod_{i=1}^d a_i, \quad R = \det B = \prod_{i=1}^d b_i. \end{aligned} \tag{12.12}$$

The Zibulski–Zeevi matrix below has R rows and N columns. Full row rank already forces $N \geq R$; the common-zero theorem will produce the stronger inequality $N \geq R + d$.

Set

$$\mathcal{I}_A = \prod_{i=1}^d \{0, 1, \dots, a_i - 1\}, \quad \mathcal{I}_B = \prod_{i=1}^d \{0, 1, \dots, b_i - 1\}, \tag{12.13}$$

so that $|\mathcal{I}_A| = N$ and $|\mathcal{I}_B| = R$. We use the base rectangle

$$Q_B = [0, 1)^d \times \prod_{i=1}^d [0, b_i^{-1}). \tag{12.14}$$

Following the vector-valued Zak representation of Zibulski–Zeevi [45], define

$$\mathcal{Z}f(x, \eta) = \left(Zf(x, \eta + \mathbf{B}^{-1}s) \right)_{s \in \mathcal{I}_\mathbf{B}} \in \mathbb{C}^R. \quad (12.15)$$

The associated $R \times N$ Zibulski–Zeevi matrix of the window is obtained by placing the vectors $\mathcal{Z}(T_{Dt}g)$ in columns:

$$\mathcal{Z}_g(x, \eta) = \left[\mathcal{Z}(T_{Dt}g)(x, \eta) \right]_{t \in \mathcal{I}_\mathbf{A}}, \quad (\mathcal{Z}_g(x, \eta))_{s,t} = Zg(x - Dt, \eta + \mathbf{B}^{-1}s). \quad (12.16)$$

The unitarity of the scalar Zak transform and the decomposition of the frequency cube into the R rectangles indexed by s show that

$$f \mapsto \mathcal{Z}f$$

is a unitary map from $L^2(\mathbb{R}^d)$ onto $L^2(Q_\mathbf{B}; \mathbb{C}^R)$. In particular,

$$\|f\|_2^2 = \int_{Q_\mathbf{B}} \|\mathcal{Z}f(x, \eta)\|_{\mathbb{C}^R}^2 dx d\eta. \quad (12.17)$$

Every $k \in \mathbb{Z}^d$ has a unique decomposition

$$k = \mathbf{A}\ell + t, \quad \ell \in \mathbb{Z}^d, \quad t \in \mathcal{I}_\mathbf{A}. \quad (12.18)$$

Since $DA = \mathbf{B}$, $Dk = \mathbf{B}\ell + Dt$. By Proposition 11.2, items (ii)–(iii),

$$\begin{aligned} Z(M_n T_{Dk}g)(x, \omega) &= e^{2\pi i \langle n, x \rangle} Zg(x - \mathbf{B}\ell - Dt, \omega) \\ &= e^{2\pi i \langle n, x \rangle} e^{-2\pi i \langle \mathbf{B}\ell, \omega \rangle} Zg(x - Dt, \omega). \end{aligned} \quad (12.19)$$

Writing $\omega = \eta + \mathbf{B}^{-1}s$ removes the residue phase, because $\langle \mathbf{B}\ell, \mathbf{B}^{-1}s \rangle = \langle \ell, s \rangle \in \mathbb{Z}$. Thus, first for $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\langle f, M_n T_{Dk}g \rangle = \int_{Q_\mathbf{B}} \left[\mathcal{Z}_g(x, \eta)^* \mathcal{Z}f(x, \eta) \right]_t e^{-2\pi i \langle n, x \rangle} e^{2\pi i \langle \mathbf{B}\ell, \eta \rangle} dx d\eta. \quad (12.20)$$

The functions

$$\sqrt{R} e^{2\pi i \langle n, x \rangle} e^{-2\pi i \langle \mathbf{B}\ell, \eta \rangle}, \quad n, \ell \in \mathbb{Z}^d, \quad (12.21)$$

form an orthonormal basis of $L^2(Q_\mathbf{B})$. Parseval's identity in (12.20), followed by summation over $t \in \mathcal{I}_\mathbf{A}$, gives

$$\sum_{n, k \in \mathbb{Z}^d} |\langle f, M_n T_{Dk}g \rangle|^2 = \frac{1}{R} \int_{Q_\mathbf{B}} \|\mathcal{Z}_g(x, \eta)^* \mathcal{Z}f(x, \eta)\|_{\mathbb{C}^N}^2 dx d\eta. \quad (12.22)$$

Since Zg is continuous, $\mathcal{Z}_g$ is bounded on the compact closure $\overline{Q_\mathbf{B}}$, and hence on $Q_\mathbf{B}$, and a density argument extends (12.22) from Schwartz functions to every $f \in L^2(\mathbb{R}^d)$.

Let A_0 and B_0 be lower and upper frame bounds, respectively, for $\mathcal{G}(g, D\mathbb{Z}^d \times \mathbb{Z}^d)$. Combining the frame inequalities with (12.17) and (12.22), and using that $f \mapsto \mathcal{Z}f$ is onto, gives, for every $F \in L^2(Q_\mathbf{B}; \mathbb{C}^R)$,

$$\begin{aligned} RA_0 \int_{Q_\mathbf{B}} \|F(x, \eta)\|_{\mathbb{C}^R}^2 dx d\eta &\leq \int_{Q_\mathbf{B}} \left\langle \mathcal{Z}_g(x, \eta) \mathcal{Z}_g(x, \eta)^* F(x, \eta), F(x, \eta) \right\rangle_{\mathbb{C}^R} dx d\eta \\ &\leq RB_0 \int_{Q_\mathbf{B}} \|F(x, \eta)\|_{\mathbb{C}^R}^2 dx d\eta. \end{aligned}$$

Fix $z \in \mathbb{C}^R$ and a measurable set $E \subset Q_\mathbf{B}$, and apply this inequality to $F = \mathbf{1}_E z$. This gives the single chain

$$RA_0 |E| \|z\|_2^2 \leq \int_E \left\langle \mathcal{Z}_g(x, \eta) \mathcal{Z}_g(x, \eta)^* z, z \right\rangle_{\mathbb{C}^R} dx d\eta \leq RB_0 |E| \|z\|_2^2.$$

Divide by $|E|$ and take a sequence of cubes E shrinking to a Lebesgue point (x, η) of the scalar integrand. For each fixed z we obtain

$$RA_0\|z\|_2^2 \leq \left\langle \mathcal{Z}_g(x, \eta) \mathcal{Z}_g(x, \eta)^* z, z \right\rangle_{\mathbb{C}^R} \leq RB_0\|z\|_2^2$$

for almost every (x, η) . Apply this first to a countable dense subset of $\mathbb{C}^R$, intersect the corresponding full-measure sets, and then pass to all $z \in \mathbb{C}^R$ by continuity in z . The resulting quadratic-form inequality is equivalent to the pointwise matrix inequality

$$RA_0 I_R \leq \mathcal{Z}_g(x, \eta) \mathcal{Z}_g(x, \eta)^* \leq RB_0 I_R \quad (12.23)$$

for almost every $(x, \eta) \in Q_B$.

We next record the boundary relations and use them to upgrade almost-everywhere full rank to everywhere full rank. For $1 \leq i \leq d$, define unitary matrices on $\mathbb{C}^{\mathcal{I}_B}$ by

$$(\Delta_i z)_s = e^{2\pi i s_i / b_i} z_s, \quad (C_i z)_s = z_{s+e_i \pmod{b_i}}. \quad (12.24)$$

They satisfy $\Delta_i^{b_i} = C_i^{b_i} = I_R$. The Zak boundary rules give

$$\mathcal{Z}_g(x + e_i, \eta) = e^{2\pi i \eta_i} \Delta_i \mathcal{Z}_g(x, \eta), \quad (12.25)$$

$$\mathcal{Z}_g(x, \eta + b_i^{-1} e_i) = C_i \mathcal{Z}_g(x, \eta). \quad (12.26)$$

Indeed, the first identity follows entry by entry from

$$Zg(x + e_i - Dt, \eta + B^{-1}s) = e^{2\pi i(\eta_i + s_i/b_i)} Zg(x - Dt, \eta + B^{-1}s),$$

while increasing η_i by $1/b_i$ cyclically increments the row label s_i . Iterating b_i times gives

$$\mathcal{Z}_g(x + b_i e_i, \eta) = e^{2\pi i b_i \eta_i} \mathcal{Z}_g(x, \eta), \quad (12.27)$$

$$\mathcal{Z}_g(x, \eta + e_i) = \mathcal{Z}_g(x, \eta). \quad (12.28)$$

The smallest eigenvalue of $\mathcal{Z}_g \mathcal{Z}_g^*$ is continuous. By (12.23), it is bounded below by RA_0 almost everywhere on the base rectangle. If it were smaller at one point, continuity would give an open set of positive measure on which the same strict inequality held, a contradiction. The boundary relations propagate the conclusion throughout $\mathbb{R}^{2d}$. Hence

$$\text{rank } \mathcal{Z}_g(x, \eta) = R, \quad (x, \eta) \in \mathbb{R}^{2d}. \quad (12.29)$$

We now introduce a diagonal gauge and derive the common-zero contradiction. For $s \in \mathcal{I}_B$ define the diagonal unitary matrix

$$\Gamma(x)_{s,s} = \exp \left(-2\pi i \sum_{j=1}^d \frac{s_j}{b_j} x_j \right). \quad (12.30)$$

Then $\Gamma(x + e_i) = \Delta_i^{-1} \Gamma(x)$. Define the $N \times R$ matrix

$$S_g(x, \eta) = \mathcal{Z}_g(x, \eta)^T \Gamma(x). \quad (12.31)$$

It is continuous and has full column rank R everywhere. Using (12.25), the identity $\Delta_i^T = \Delta_i$, and the boundary rule for Γ , we obtain

$$S_g(x + e_i, \eta) = e^{2\pi i \eta_i} S_g(x, \eta), \quad (12.32)$$

$$S_g(x, \eta + e_i) = S_g(x, \eta). \quad (12.33)$$

Equivalently,

$$S_g(x + n, \eta + m) = e^{2\pi i \langle n, \eta \rangle} S_g(x, \eta), \quad n, m \in \mathbb{Z}^d. \quad (12.34)$$

Thus S_g satisfies the hypotheses of Theorem 11.4, which gives $N \geq R + d$. $\square$

This is the diagonal continuous-Zak assertion used in Theorem 1.7(d).

13. SYMPLECTIC AND METAPLECTIC EXTENSION MACHINERY

We now prepare the passage from the diagonal model to arbitrary phase-space lattices, using the symplectic form, adjoint lattice, symplectic rationality, and the invariant ν introduced in Subsection 1.2. Let

$$J = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix},$$

so that $\sigma(z, w) = z^T J w$. The symplectic group is

$$\mathrm{Sp}(2d, \mathbb{R}) = \{S \in \mathrm{GL}_{2d}(\mathbb{R}) : S^T J S = J\}.$$

Thus S is symplectic exactly when $\sigma(Sz, Sw) = \sigma(z, w)$ for all $z, w \in \mathbb{R}^{2d}$. Two phase-space lattices Λ and Λ' are *symplectically equivalent* if $\Lambda' = S\Lambda$ for some $S \in \mathrm{Sp}(2d, \mathbb{R})$. In concrete terms, the phase-space variable $z = (x, \omega)$ is replaced by $Sz = (x', \omega')$, and every lattice point λ is replaced by $S\lambda$. Symplectic equivalence preserves covolume and satisfies

$$(S\Lambda)^\circ = S\Lambda^\circ.$$

Write a real $2d \times 2d$ matrix in blocks as

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A, B, C, D \in M_d(\mathbb{R}).$$

The block criterion for symplectic matrices [25, Lemma 9.4.1(c)] states that $S \in \mathrm{Sp}(2d, \mathbb{R})$ if and only if

$$A^T C = C^T A, \quad B^T D = D^T B, \quad A^T D - C^T B = I_d. \quad (13.1)$$

Equivalently,

$$AB^T = BA^T, \quad CD^T = DC^T, \quad AD^T - BC^T = I_d. \quad (13.2)$$

For example, if A is invertible, these identities imply that CA^{-1} and $A^{-1}B$ are symmetric. These elementary block relations will be used below to read support properties directly from a symplectic matrix.

The group $\mathrm{Sp}(2d, \mathbb{R})$ is generated by matrices of the following three forms:

$$\begin{aligned} S_A &= \begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix}, & A &\in \mathrm{GL}_d(\mathbb{R}), \\ V_B &= \begin{pmatrix} I_d & 0 \\ B & I_d \end{pmatrix}, & B &\in M_d(\mathbb{R}), \quad B = B^T, \\ J &= \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}. \end{aligned}$$

The matrix S_A is called the *symplectic dilation associated with A* ; it acts by

$$(x, \omega) \mapsto (Ax, A^{-T}\omega).$$

The metaplectic group is a double cover of $\mathrm{Sp}(2d, \mathbb{R})$ by unitary operators on $L^2(\mathbb{R}^d)$. In fact, for each $S \in \mathrm{Sp}(2d, \mathbb{R})$ there exists a unitary *metaplectic lift* $\mu(S)$. Up to the sign ambiguity arising from the double cover, we have $\mu(S_1 S_2) = \mu(S_1) \mu(S_2)$. With the time-frequency convention used here, a metaplectic lift satisfies

$$\mu(S) \pi(z) \mu(S)^{-1} = c(S, z) \pi(Sz), \quad |c(S, z)| = 1.$$

Up to constants of modulus one, convenient lifts of the three generators are

$$(\mu(S_A)f)(t) = |\det A|^{-1/2} f(A^{-1}t), \quad (\mu(V_B)f)(t) = e^{\pi i t^T B t} f(t),$$

and, with the Fourier convention used in this paper,

$$(\mu(J)f)(\omega) = \int_{\mathbb{R}^d} f(t) e^{-2\pi i \langle t, \omega \rangle} dt = \widehat{f}(\omega).$$

Consequently, $\mu(S)$ is unitary on $L^2(\mathbb{R}^d)$, maps $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}_0(\mathbb{R}^d)$ onto themselves, and preserves frame bounds when the lattice and all windows are transformed together. In particular,

$$\mathcal{G}(g, \Lambda) \text{ is Parseval} \iff \mathcal{G}(\mu(S)g, S\Lambda) \text{ is Parseval.}$$

These standard facts may be found in [19], [25, Section 9.4], and [22].

We next isolate the arithmetic notion relevant to the normal form. For a full-rank phase-space lattice Λ , recall

$$\Lambda_{\text{int}} = \Lambda \cap \Lambda^\circ.$$

The lattice is *symplectically rational* if $[\Lambda : \Lambda_{\text{int}}] < \infty$. A real matrix is called *entrywise rational* if all of its entries lie in $\mathbb{Q}$.

Lemma. *Let $\Lambda = M\mathbb{Z}^{2d}$ and put $\Theta_\Lambda = M^T J M$. Then Λ is symplectically rational if and only if Θ_Λ is entrywise rational. In particular, let*

$$\Lambda = \Gamma \times \Phi, \quad \Gamma = A_0 \mathbb{Z}^d, \quad \Phi = B_0 \mathbb{Z}^d,$$

with $A_0, B_0 \in GL_d(\mathbb{R})$. The following conditions are equivalent:

- (i) $\Gamma \times \Phi$ is symplectically rational;
- (ii) the pairing matrix $A_0^T B_0$ is entrywise rational;
- (iii) the Euclidean lattices Γ and Φ^* are commensurable.

Proof. Suppose first that Θ_Λ is entrywise rational. Choose $q \in \mathbb{N}$ such that $q\Theta_\Lambda$ is integral. If $\lambda = Mk$ and $\mu = M\ell$, with $k, \ell \in \mathbb{Z}^{2d}$, then

$$\sigma(q\lambda, \mu) = qk^T \Theta_\Lambda \ell \in \mathbb{Z}.$$

Thus $q\Lambda \subset \Lambda_{\text{int}}$, so $[\Lambda : \Lambda_{\text{int}}] < \infty$.

Conversely, suppose $\Lambda/\Lambda_{\text{int}}$ is finite, and let q be its exponent. Then $q\Lambda \subset \Lambda_{\text{int}}$. Applying this to the basis vectors Me_j gives

$$q(\Theta_\Lambda)_{jk} = \sigma(qMe_j, Me_k) \in \mathbb{Z}, \quad 1 \leq j, k \leq 2d.$$

Hence Θ_Λ is entrywise rational.

For the separable lattice $\Gamma \times \Phi$, the symplectic Gram matrix of the basis $\text{diag}(A_0, B_0)$ is

$$\begin{pmatrix} 0 & A_0^T B_0 \\ -B_0^T A_0 & 0 \end{pmatrix}.$$

This proves the equivalence of (i) and (ii). Under the common linear change of Euclidean coordinates A_0^{-1} , the pair (Γ, Φ^*) becomes

$$(\mathbb{Z}^d, (A_0^T B_0)^{-T} \mathbb{Z}^d).$$

A lattice $C\mathbb{Z}^d$ is commensurable with $\mathbb{Z}^d$ exactly when $C \in GL_d(\mathbb{Q})$. This proves the equivalence with (iii). $\square$

We use the ordinary Smith normal form and its alternating, or skew, analogue in the proof of the normal-form theorem.

Theorem (Smith–Newman [40]). *Let $K \in M_d(\mathbb{Z})$ have nonzero determinant. There exist $U, V \in GL_d(\mathbb{Z})$ and positive integers $s_1, \dots, s_d$, with s_j dividing s_{j+1} for $j = 1, \dots, d-1$, such that*

$$U^T K V = \text{diag}(s_1, \dots, s_d).$$

Let $\Theta \in M_{2d}(\mathbb{Z})$ be nonsingular and alternating, that is $\Theta^T = -\Theta$. There exist $W \in GL_{2d}(\mathbb{Z})$ and positive integers $h_1, \dots, h_d$, with h_j dividing h_{j+1} for $j = 1, \dots, d-1$, such that

$$W^T \Theta W = \begin{pmatrix} 0 & H \\ -H & 0 \end{pmatrix}, \quad H = \text{diag}(h_1, \dots, h_d).$$

See [40, Chapters II and IV]; the alternating statement is the skew Smith normal form.

Theorem 13.1. *Every symplectically rational full-rank phase-space lattice $\Lambda \subset \mathbb{R}^{2d}$ is symplectically equivalent to*

$$\Lambda_D = D\mathbb{Z}^d \times \mathbb{Z}^d, \quad D = \text{diag}\left(\frac{b_1}{a_1}, \dots, \frac{b_d}{a_d}\right), \quad (a_i, b_i) = 1, \quad (13.3)$$

where all a_i, b_i are positive integers. Thus there exists $S \in \text{Sp}(2d, \mathbb{R})$ such that $\Lambda = S\Lambda_D$.

If

$$N = \prod_{i=1}^d a_i, \quad R = \prod_{i=1}^d b_i,$$

then

$$\nu(\Lambda) = \nu(\Lambda_D) = N, \quad \nu(\Lambda^\circ) = \nu(\Lambda_D^\circ) = R, \quad \text{covol}(\Lambda) = \text{covol}(\Lambda_D) = \frac{R}{N}. \quad (13.4)$$

Consequently, the symplectically rational conditions (SG) and mSG_q become, respectively,

$$N \geq R + d, \quad qN \geq R + d.$$

If $\Lambda = \Gamma \times \Phi$ is separable, the normalizing symplectic matrix may be chosen to be a symplectic dilation. If $\Gamma + \Phi^*$ is a lattice, then

$$\nu(\Gamma \times \Phi) = [\Gamma + \Phi^* : \Phi^*], \quad \nu((\Gamma \times \Phi)^\circ) = [\Gamma + \Phi^* : \Gamma].$$

Proof. We begin with the separable case. Write

$$\Gamma = A_0\mathbb{Z}^d, \quad \Phi = B_0\mathbb{Z}^d, \quad C_{\Gamma, \Phi} = A_0^T B_0.$$

By the preceding lemma, $C_{\Gamma, \Phi}$ is entrywise rational. Choose $m \in \mathbb{N}$ so that $mC_{\Gamma, \Phi}$ is integral. The first part of the preceding Smith–Newman theorem gives $U, V \in GL_d(\mathbb{Z})$ and positive integers $s_1, \dots, s_d$, with s_j dividing s_{j+1} for $j = 1, \dots, d-1$, such that

$$U^T(mC_{\Gamma, \Phi})V = \text{diag}(s_1, \dots, s_d).$$

Put

$$D_0 = \text{diag}(s_1/m, \dots, s_d/m).$$

The symplectic dilation associated with A_0^{-1} sends $\Gamma \times \Phi$ to $\mathbb{Z}^d \times C_{\Gamma, \Phi}\mathbb{Z}^d$. The symplectic dilation associated with U^{-1} then sends this lattice to

$$\mathbb{Z}^d \times U^T C_{\Gamma, \Phi} \mathbb{Z}^d = \mathbb{Z}^d \times D_0 \mathbb{Z}^d,$$

because $V\mathbb{Z}^d = \mathbb{Z}^d$. Finally, the symplectic dilation associated with D_0 sends this lattice to

$$D_0 \mathbb{Z}^d \times \mathbb{Z}^d.$$

Writing each diagonal entry of D_0 in lowest terms as b_i/a_i gives (13.3). The product of the three symplectic dilations is again a symplectic dilation. Thus no chirp or Fourier transform is needed in the separable reduction.

We now treat an arbitrary symplectically rational phase-space lattice. Choose a basis matrix $M \in GL_{2d}(\mathbb{R})$ with $\Lambda = M\mathbb{Z}^{2d}$ and set $\Theta_\Lambda = M^T J M$. By the preceding lemma, Θ_Λ is entrywise rational. Choose $q \in \mathbb{N}$ such that $q\Theta_\Lambda$ is integral. The second part of the preceding Smith–Newman theorem, applied to the invertible and alternating matrix $q\Theta_\Lambda$, gives $U \in GL_{2d}(\mathbb{Z})$ and positive integers $h_1, \dots, h_d$, with h_j dividing h_{j+1} for $j = 1, \dots, d-1$, such that

$$U^T \Theta_\Lambda U = \begin{pmatrix} 0 & D_0 \\ -D_0 & 0 \end{pmatrix}, \quad D_0 = \text{diag}\left(\frac{h_1}{q}, \dots, \frac{h_d}{q}\right).$$

Let

$$C_0 = \begin{pmatrix} D_0 & 0 \\ 0 & I_d \end{pmatrix}, \quad S = MUC_0^{-1}.$$

Since

$$C_0^T J C_0 = \begin{pmatrix} 0 & D_0 \\ -D_0 & 0 \end{pmatrix} = U^T M^T J M U,$$

we obtain

$$S^T J S = C_0^{-T} U^T M^T J M U C_0^{-1} = J.$$

Thus S is symplectic and

$$\Lambda = M\mathbb{Z}^{2d} = MU\mathbb{Z}^{2d} = SC_0\mathbb{Z}^{2d} = S(D_0\mathbb{Z}^d \times \mathbb{Z}^d).$$

Reducing the diagonal entries of D_0 gives the coprime pairs (a_i, b_i) in (13.3).

For the diagonal lattice, the three formulas in (13.4) were computed explicitly in Lemma 12.1. If $\Lambda = S\Lambda_D$, then $(S\Lambda_D)^\circ = S\Lambda_D^\circ$, $S(\Lambda_D \cap \Lambda_D^\circ) = \Lambda \cap \Lambda^\circ$, and $\det S = 1$. Therefore the two indices and the covolume are unchanged, proving all three equalities in (13.4).

In the separable case, the common linear changes above carry both Γ and Φ^* by the same invertible matrix, so the indices in their sum are unchanged. On the diagonal representative the two indices are N and R , which gives

$$\nu(\Gamma \times \Phi) = [\Gamma + \Phi^* : \Phi^*], \quad \nu((\Gamma \times \Phi)^\circ) = [\Gamma + \Phi^* : \Gamma].$$

□

Proof of Proposition 1.6. By the separable part of Theorem 13.1, $\Gamma \times \Phi$ is symplectically rational exactly when $\Gamma + \Phi^*$ is a lattice. If $\Gamma + \Phi^*$ is not discrete, then

$$\text{covol}(\Gamma \times \Phi) < 1 \iff \text{covol}(\Gamma) < \text{covol}(\Phi^*),$$

which is the non-discrete branch of (TeP). If $\Gamma + \Phi^*$ is a lattice, the index formulas in the theorem show that the symplectically rational branch of (SG) is precisely

$$[\Gamma + \Phi^* : \Phi^*] \geq [\Gamma + \Phi^* : \Gamma] + d,$$

which is the commensurable branch of (TeP). □

13.1. Effect on the support of the window

The distinction between separable and general lattices is important for the effect on the support of the window function g . The block criterion gives a precise support-preserving case. Let

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2d, \mathbb{R}).$$

If $B = 0$, then (13.1) gives $D = A^{-T}$ and shows that

$$Q = CA^{-1}$$

is symmetric. Hence

$$S = \begin{pmatrix} I_d & 0 \\ Q & I_d \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix},$$

and, up to a constant of modulus one,

$$(\mu(S)f)(t) = |\det A|^{-1/2} e^{\pi i t^T Q t} f(A^{-1}t). \quad (13.5)$$

Consequently,

$$\text{supp } \mu(S)f = A(\text{supp } f). \quad (13.6)$$

Thus $\mu(S)$ maps $C_c^\infty(\mathbb{R}^d)$ onto itself whenever $B = 0$. Conversely, suppose $B \neq 0$ and let $r = \text{rank } B \geq 1$. In the standard Bruhat factorization of S , all factors except one preserve compact support: they are invertible linear changes of variables or multiplication by chirps. The remaining factor is an r -dimensional partial Fourier transform. Write the variables for that factor as $(x, y) \in \mathbb{R}^r \times \mathbb{R}^{d-r}$. If both its input f_0 and output were compactly supported, then Fubini's theorem would give, for almost every fixed y , a compactly supported slice $x \mapsto f_0(x, y)$ whose Fourier transform in x is also compactly supported. That

Fourier transform extends to an entire function of $\zeta \in \mathbb{C}^r$ by the Paley–Wiener theorem. It vanishes on an open rectangle in the complement of its real support; applying the one-variable identity theorem successively in $\zeta_1, \dots, \zeta_r$ shows that it is identically zero. Hence $f_0(\cdot, y) = 0$ for almost every y , and therefore $f_0 = 0$. Undoing the support-preserving factors gives the same conclusion for the original input. Thus a nonzero compactly supported function cannot be carried to a compactly supported function when $B \neq 0$, and $B = 0$ is the intrinsic block criterion for a metaplectic operator to preserve compact support.

The Fourier transform of $\mu(S)f$ is, up to a constant of modulus one, $\mu(JS)f$. Since

$$JS = \begin{pmatrix} C & D \\ -A & -B \end{pmatrix},$$

Equation (13.5) applies to JS exactly when $D = 0$. Thus $f \in C_c^\infty(\mathbb{R}^d)$ implies $\widehat{\mu(S)f} \in C_c^\infty(\mathbb{R}^d)$ when $D = 0$.

In the separable part of Theorem 13.1, the normalizing matrix is a symplectic dilation, so its upper-right block is zero. The reduction of a separable lattice to diagonal form therefore does not destroy compact support. For an arbitrary symplectically rational phase-space lattice, the normal-form theorem produces a general symplectic matrix, and its upper-right block need not vanish. This is why transfer to a general lattice naturally guarantees a Schwartz or Feichtinger-algebra window, but not a compactly supported one.

14. EXTENSION FROM THE DIAGONAL MODEL TO ARBITRARY LATTICES

We now transfer the existence and necessity conclusions from the diagonal model in the order in which they were proved: first necessity, then existence.

14.1. Extension of the necessity result (proof of Theorem 1.7(e))

Assume first that Λ is symplectically rational and that $\mathcal{G}(g, \Lambda)$ is a frame for some $g \in \mathcal{S}(\mathbb{R}^d)$. By Theorem 13.1, there are $S \in \text{Sp}(2d, \mathbb{R})$ and a diagonal D as in (13.3) such that

$$\Lambda = S(D\mathbb{Z}^d \times \mathbb{Z}^d). \quad (14.1)$$

The reduced window $g_0 = \mu(S)^{-1}g$ is Schwartz, hence belongs to $C_Z(\mathbb{R}^d)$, and

$$\mathcal{G}(g_0, D\mathbb{Z}^d \times \mathbb{Z}^d)$$

is a frame. Theorem 12.3 gives

$$N \geq R + d. \quad (14.2)$$

By (13.4) and symplectic invariance, this is

$$v(\Lambda) \geq v(\Lambda^\circ) + d. \quad (14.3)$$

Thus the symplectically rational inequality in (SG) is necessary.

Theorem 14.1. *Let $\Lambda \subset \mathbb{R}^{2d}$ be symplectically rational and choose a diagonalization*

$$\Lambda = S(D\mathbb{Z}^d \times \mathbb{Z}^d).$$

If $\mathcal{G}(g, \Lambda)$ is a frame and $Z(\mu(S)^{-1}g)$ is continuous, then

$$v(\Lambda) \geq v(\Lambda^\circ) + d.$$

For the diagonal representative this is

$$\prod_{i=1}^d a_i \geq \prod_{i=1}^d b_i + d.$$

Proof. The transformed window belongs to $C_Z(\mathbb{R}^d)$ and generates a frame on the diagonal representative. Apply Theorem 12.3 and then Theorem 13.1. $\square$

Corollary 14.2. *Let $\Lambda \subset \mathbb{R}^{2d}$ be symplectically rational. If $g \in S_0(\mathbb{R}^d)$ and $\mathcal{G}(g, \Lambda)$ is a frame, then*

$$\text{covol}(\Lambda) < 1, \quad \nu(\Lambda) \geq \nu(\Lambda^\circ) + d.$$

Proof. Choose a diagonalization

$$\Lambda = S(D\mathbb{Z}^d \times \mathbb{Z}^d).$$

Metaplectic invariance of the Feichtinger algebra gives $\mu(S)^{-1}g \in S_0(\mathbb{R}^d)$. By Proposition 11.2(v), its Zak transform is continuous. Theorem 14.1 gives the (SG) inequality. $\square$

Remark 14.3. Theorem 14.1 concerns the Zak transform in the coordinates adapted to a chosen diagonal representative. Its conclusion is that every frame window has discontinuous *adapted* Zak transform when the symplectically rational (SG) inequality fails. This does not imply that the standard Zak transform Zg is discontinuous: continuity of $Z(\mu(S)^{-1}g)$ and continuity of Zg are not equivalent for a general metaplectic change of coordinates. The diagonal statement in Theorem 1.7(d) therefore uses the larger class C_Z . The coordinate-free statement in part (e) uses the smaller but metaplectically invariant Wiener amalgam $W(\mathcal{FL}^1, L^1) = S_0$.

Suppose next that Λ is not symplectically rational and that $\mathcal{G}(g, \Lambda)$ is a frame with $g \in S_0(\mathbb{R}^d)$. The density theorem gives $\text{covol}(\Lambda) \leq 1$. Equality is impossible already in the Feichtinger algebra. Indeed, frames with an S_0 window are stable under sufficiently small linear deformations of the lattice [18, 28]. If $\text{covol}(\Lambda) = 1$, then for every sufficiently small $\delta > 0$ the system

$$\mathcal{G}(g, (1 + \delta)\Lambda)$$

would still be a frame, while

$$\text{covol}((1 + \delta)\Lambda) = (1 + \delta)^{2d} \text{covol}(\Lambda) > 1,$$

contradicting the density theorem. Therefore

$$\text{covol}(\Lambda) < 1. \tag{14.4}$$

This proves the necessity of the remaining inequality in (SG) for S_0 windows, and hence in particular for Schwartz windows.

14.2. Existence under the condition (SG) (proof of Theorem 1.7(a)–(c))

Assume first that Λ is symplectically rational and that the inequalities in (SG) hold. Write

$$\Lambda = S(D\mathbb{Z}^d \times \mathbb{Z}^d)$$

as in Theorem 13.1. Then $N \geq R + d$, so Theorem 12.2 provides a $g_0 \in C_c^\infty(\mathbb{R}^d)$ whose Gabor system on the diagonal lattice is Parseval. Metaplectic covariance shows that

$$g = \mu(S)g_0 \in \mathcal{S}(\mathbb{R}^d)$$

generates a Parseval frame on Λ .

For a separable lattice $\Lambda = \Gamma \times \Phi$, one obtains more. Proposition 1.6 identifies the inequalities in (SG) with those in (TeP) for the pair (Γ, Φ^*) . The construction of Part A and the mollification and energy normalization carried out in Section 12.1 use only this pair. Therefore the same argument gives the following statement, whether or not the separable lattice is symplectically rational.

Theorem 14.4. *Let $\Gamma, \Phi \subset \mathbb{R}^d$ be full-rank lattices. Suppose*

$$\text{covol}(\Gamma) < \text{covol}(\Phi^*) = \text{covol}(\Phi)^{-1}$$

and, if $\Gamma + \Phi^$ is discrete, suppose also that*

$$[\Gamma + \Phi^* : \Phi^*] \geq [\Gamma + \Phi^* : \Gamma] + d.$$

Then there exists a nonnegative $g \in C_c^\infty(\mathbb{R}^d)$ such that $\mathcal{G}(g, \Gamma \times \Phi)$ is a Parseval frame and

$$\text{supp } g \cap (\text{supp } g + \xi^*) = \emptyset, \quad \xi^* \in \Phi^* \setminus \{0\}, \quad (14.5)$$

$$\sum_{\gamma \in \Gamma} |g(x + \gamma)|^2 = \text{covol}(\Phi), \quad x \in \mathbb{R}^d. \quad (14.6)$$

Equivalently, $\text{covol}(\Phi)^{-1}|g|^2$ is a smooth Γ -partition of unity. In particular,

$$g(x)g(x - \xi^*) = 0, \quad x \in \mathbb{R}^d, \quad \xi^* \in \Phi^* \setminus \{0\}.$$

Alternatively, one may choose $g \in \mathcal{S}(\mathbb{R}^d)$ so that $\widehat{g} \in C_c^\infty(\mathbb{R}^d)$ is nonnegative and satisfies

$$\text{supp } \widehat{g} \cap (\text{supp } \widehat{g} + \gamma^*) = \emptyset, \quad \gamma^* \in \Gamma^* \setminus \{0\}, \quad (14.7)$$

$$\sum_{\phi \in \Phi} |\widehat{g}(\omega + \phi)|^2 = \text{covol}(\Gamma), \quad \omega \in \mathbb{R}^d. \quad (14.8)$$

Equivalently, $\text{covol}(\Gamma)^{-1}|\widehat{g}|^2$ is a smooth Φ -partition of unity. In the time-side statement, one may alternatively choose a nonnegative $g_{\text{pa}} \in C_c(\mathbb{R}^d)$ such that $\text{covol}(\Phi)^{-1}g_{\text{pa}}^2$ is a continuous piecewise-affine Γ -partition of unity. In the Fourier-side statement one may alternatively arrange that $\widehat{g} \in C_c(\mathbb{R}^d)$ and $\text{covol}(\Gamma)^{-1}|\widehat{g}|^2$ is a continuous piecewise-affine Φ -partition of unity.

Proof. Proposition 1.6 shows that the pair (Γ, Φ^*) satisfies the two inequalities in (TeP). Corollary 9.6 gives a nonnegative compactly supported piecewise-affine Γ -partition of unity φ with an actual Φ^* -support buffer. Hence

$$g_{\text{pa}} = \sqrt{\text{covol}(\Phi)} \varphi$$

is compactly supported and $\text{covol}(\Phi)^{-1}g_{\text{pa}}^2 = \varphi$ is the piecewise-affine Γ -partition of unity. It satisfies (14.5)–(14.6). The adjoint-lattice argument below proves that its Gabor system is Parseval. Mollification and energy normalization as in Section 12.1 instead give a nonnegative $g \in C_c^\infty(\mathbb{R}^d)$ satisfying the same identities.

Let

$$h_0 = \text{covol}(\Gamma \times \Phi)^{-1/2} g.$$

Then

$$\sum_{\gamma \in \Gamma} |h_0(x + \gamma)|^2 = \frac{1}{\text{covol}(\Gamma)}, \quad x \in \mathbb{R}^d.$$

The adjoint lattice is $(\Gamma \times \Phi)^\circ = \Phi^* \times \Gamma^*$. Distinct translations of h_0 by elements of Φ^* have disjoint supports. For a fixed translation and $\gamma^*, \gamma'^* \in \Gamma^*$, put $\eta = \gamma'^* - \gamma^*$. Periodization over a fundamental domain F_Γ gives

$$\begin{aligned} \langle h_0, M_\eta h_0 \rangle &= \int_{F_\Gamma} \sum_{\gamma \in \Gamma} |h_0(x + \gamma)|^2 e^{2\pi i \langle \eta, x + \gamma \rangle} dx \\ &= \frac{1}{\text{covol}(\Gamma)} \int_{F_\Gamma} e^{2\pi i \langle \eta, x \rangle} dx = \delta_{\gamma^*, \gamma'^*}. \end{aligned}$$

Thus $\mathcal{G}(h_0, \Phi^* \times \Gamma^*)$ is an orthonormal system. The normalized duality principle, Theorem 11.1, now implies that $\mathcal{G}(g, \Gamma \times \Phi)$ is a Parseval frame.

For the Fourier-side alternative, apply the result just proved to the separable lattice

$$J(\Gamma \times \Phi) = \Phi \times (-\Gamma).$$

The hypotheses are unchanged because J is symplectic. We obtain a nonnegative $h \in C_c^\infty(\mathbb{R}^d)$ such that $\mathcal{G}(h, \Phi \times (-\Gamma))$ is Parseval and

$$\text{supp } h \cap (\text{supp } h + \gamma^*) = \emptyset, \quad \gamma^* \in \Gamma^* \setminus \{0\},$$

$$\sum_{\phi \in \Phi} |h(\omega + \phi)|^2 = \text{covol}(\Gamma), \quad \omega \in \mathbb{R}^d.$$

Set $g = \mu(-J)h$. Then $\widehat{g} = \mu(J)g = h$, and metaplectic covariance transfers the Parseval frame from $\Phi \times (-\Gamma)$ back to $\Gamma \times \Phi$. This proves (14.7)–(14.8). If the unsmoothed precursor is used on the rotated lattice instead, the same argument gives the stated alternative with $|\widehat{g}|^2$ piecewise affine. $\square$

Proposition 14.5. *Let $a > 0$, let*

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2d, \mathbb{R}), \quad \Lambda = aSZ^{2d},$$

and assume the two inequalities in (SG) hold for Λ . If $A = 0$ or $B = 0$, then Λ admits a Parseval frame window $g \in C_c^\infty(\mathbb{R}^d)$; alternatively, it admits a compactly supported continuous Parseval window with $|g|^2$ piecewise affine. If $C = 0$ or $D = 0$, then Λ admits a Parseval frame window g such that $\widehat{g} \in C_c^\infty(\mathbb{R}^d)$; alternatively, one may arrange that $\widehat{g}$ is compactly supported and continuous and $|\widehat{g}|^2$ is piecewise affine.

Proof. The two inequalities in (SG) are symplectically invariant, so the separable lattice $a\mathbb{Z}^d \times a\mathbb{Z}^d$ satisfies the hypotheses of Theorem 14.4. Choose $h \in C_c^\infty(\mathbb{R}^d)$ so that $\mathcal{G}(h, a\mathbb{Z}^{2d})$ is Parseval. Independently, choose a compactly supported continuous Parseval window h_{pa} such that $|h_{\text{pa}}|^2$ is piecewise affine.

If $B = 0$, put $g = \mu(S)h$. Metaplectic covariance gives a Parseval frame on Λ , and Equation (13.5) gives $g \in C_c^\infty(\mathbb{R}^d)$. Applying the same operator to h_{pa} gives a compactly supported Parseval window g_{pa} with

$$|g_{\text{pa}}(t)|^2 = |\det A|^{-1} |h_{\text{pa}}(A^{-1}t)|^2,$$

which is piecewise affine. If $A = 0$, use instead

$$\Lambda = aSJ^{-1}\mathbb{Z}^{2d}.$$

The upper-right block of SJ^{-1} is $-A = 0$, so $g = \mu(SJ^{-1})h$ is compactly supported and generates a Parseval frame on Λ ; applying $\mu(SJ^{-1})$ to h_{pa} gives the corresponding modulus-square piecewise-affine alternative.

If $D = 0$, put $g = \mu(S)h$. Since the upper-right block of JS is $D = 0$,

$$\widehat{g} = \mu(J)\mu(S)h = c \mu(JS)h, \quad |c| = 1,$$

belongs to $C_c^\infty(\mathbb{R}^d)$. Applying the same expression to h_{pa} shows that $|\widehat{g}|^2$ is piecewise affine. Finally, if $C = 0$, use the representation $\Lambda = aSJ^{-1}\mathbb{Z}^{2d}$ and put $g = \mu(SJ^{-1})h$. The upper-right block of JSJ^{-1} is $-C = 0$, and the same argument gives $\widehat{g} \in C_c^\infty(\mathbb{R}^d)$, with the analogous modulus-square piecewise-affine alternative obtained from h_{pa} . $\square$

Finally, assume that Λ is not symplectically rational and $\text{covol}(\Lambda) < 1$. In the notation of Enstad–Thiel–Vilalta the correction term is zero, so their Theorem C [16] gives a one-window Schwartz Gabor frame on Λ . If S_g is its frame operator, the canonical Parseval window is $S_g^{-1/2}g$. Spectral invariance of the smooth twisted group algebra keeps this window in the Schwartz class; see [32, 39].

The necessity and existence arguments prove Theorem 1.3. They also prove the five parts of Theorem 1.7: part (a) is the existence result just established; part (b) is Theorem 14.4; part (c) is Proposition 14.5; part (d) is Theorem 12.3; and part (e) combines Corollary 14.2 with the strict-density argument for lattices that are not symplectically rational.

15. THE MULTIPLE-WINDOW EXTENSION

The paper remains focused on one-window frames. We record the multiple-window extension after completing the one-window theorem because its necessity proof uses the same diagonal matrix argument with a larger number of columns.

Multiple windows only increase the number of Zibulski–Zeevi columns. The following is the continuous-Zak necessity statement in the diagonal coordinates.

Proposition 15.1. *Retain the diagonal data in (12.12), and let $\mathbf{g} = (g_1, \dots, g_q)$ with $g_j \in C_Z(\mathbb{R}^d)$. If*

$$\mathcal{G}(\mathbf{g}, D\mathbb{Z}^d \times \mathbb{Z}^d)$$

is a frame, then

$$qN \geq R + d. \quad (15.1)$$

Proof. For each j , let $\mathcal{Z}_{g_j}(x, \eta) \in M_{R,N}(\mathbb{C})$ be the Zibulski–Zeevi matrix from (12.16), and form the horizontal concatenation

$$\mathcal{Z}_{\mathbf{g}}(x, \eta) = [\mathcal{Z}_{g_1}(x, \eta) \ \cdots \ \mathcal{Z}_{g_q}(x, \eta)] \in M_{R,qN}(\mathbb{C}). \quad (15.2)$$

Summing (12.22) over the windows gives

$$\begin{aligned} & \sum_{j=1}^q \sum_{n,k \in \mathbb{Z}^d} |\langle f, M_n T_{Dk} g_j \rangle|^2 \\ &= \frac{1}{R} \int_{Q_B} \|\mathcal{Z}_{\mathbf{g}}(x, \eta)^* \mathcal{Z}f(x, \eta)\|_{\mathbb{C}^{qN}}^2 dx d\eta. \end{aligned} \quad (15.3)$$

Consequently, a multiwindow frame with bounds A_0, B_0 is equivalent to

$$RA_0 I_R \leq \mathcal{Z}_{\mathbf{g}}(x, \eta) \mathcal{Z}_{\mathbf{g}}(x, \eta)^* \leq RB_0 I_R \quad (15.4)$$

for almost every point of the base rectangle. Since all $\mathcal{Z}_{g_j}$ are continuous, the smallest eigenvalue in (15.4) is continuous. The open-set argument in the proof of Theorem 12.3 yields

$$\text{rank } \mathcal{Z}_{\mathbf{g}}(x, \eta) = R \quad \text{for every } (x, \eta). \quad (15.5)$$

Every block satisfies the same boundary relations. With the diagonal gauge $\Gamma(x)$ from (12.30), set

$$S_{\mathbf{g}}(x, \eta) = \mathcal{Z}_{\mathbf{g}}(x, \eta)^T \Gamma(x) \in M_{qN,R}(\mathbb{C}). \quad (15.6)$$

Then

$$S_{\mathbf{g}}(x + n, \eta + m) = e^{2\pi i \langle n, \eta \rangle} S_{\mathbf{g}}(x, \eta), \quad n, m \in \mathbb{Z}^d, \quad (15.7)$$

and (15.5) says that this matrix has full column rank everywhere. Theorem 11.4, with N replaced by qN , gives $qN \geq R + d$. $\square$

Proof of Theorem 1.8. We first treat symplectically rational lattices. Suppose that $\mathcal{G}(\mathbf{g}, \Lambda)$ is a q -window frame with $g_1, \dots, g_q \in S_0(\mathbb{R}^d)$. By Theorem 13.1 and metaplectic covariance, we may work on $D\mathbb{Z}^d \times \mathbb{Z}^d$. Metaplectic invariance gives reduced windows in $S_0(\mathbb{R}^d)$, whose Zak transforms are continuous by Proposition 11.2(v), and

$$N = v(\Lambda), \quad R = v(\Lambda^\circ).$$

Proposition 15.1 gives

$$qN \geq R + d,$$

which is equivalent to

$$q v(\Lambda) \geq v(\Lambda^\circ) + d.$$

This proves necessity in the symplectically rational case for Feichtinger-algebra windows, including the stronger continuous-Zak assertion in diagonal coordinates.

For sufficiency in the symplectically rational case, put $n = v(\Lambda)$ and $m = v(\Lambda^\circ)$. Then

$$\text{covol}(\Lambda) = \frac{m}{n}.$$

Theorem D of Enstad–Thiel–Vilalta [16] supplies a Schwartz multiwindow frame with

$$\begin{aligned} k &= \left\lfloor \text{covol}(\Lambda) + \frac{d-1}{v(\Lambda)} \right\rfloor + 1 \\ &= \left\lfloor \frac{m+d-1}{n} \right\rfloor + 1 = \left\lceil \frac{m+d}{n} \right\rceil \end{aligned} \quad (15.8)$$

windows. The symplectically rational condition mSG_q is exactly $q \geq k$.

Now let Λ be a lattice that is not symplectically rational. Necessity begins with the standard multiwindow density theorem,

$$\text{covol}(\Lambda) \leq q.$$

Equality cannot occur for Feichtinger-algebra windows by the multiwindow amalgam Balian–Low theorem; see Enstad [15, Theorems 5.4–5.6] for multiwindow density, Gerhold–Lamando–Luef [21, Theorem 3.38 and Corollary 3.39] for the multiwindow deformation theorem and the critical-density obstruction, and Zibulski–Zeevi [45] for the classical rational multiwindow framework. Hence

$$\text{covol}(\Lambda) < q,$$

which is the corresponding branch of mSG_q .

Conversely, when Λ is not symplectically rational, Theorem D of Enstad–Thiel–Vilalta gives a Schwartz frame with

$$k = \lfloor \text{covol}(\Lambda) \rfloor + 1$$

windows. The inequality $\text{covol}(\Lambda) < q$ is equivalent to $q \geq k$. Thus in both the symplectically rational case and the case of lattices that are not symplectically rational, we obtain a Schwartz frame with at most q windows. If the construction produces $k < q$ windows, set $\ell = q - k + 1$ and replace one nonzero window h by ℓ copies of $h/\sqrt{\ell}$. The total number of windows is then $(k-1) + \ell = q$, and the sum of the copied windows' frame-operator contributions equals that of h . Finally apply the inverse square root of the common frame operator to every window. Spectral invariance keeps all canonical Parseval windows in the Schwartz class. This proves sufficiency and produces exactly q windows.

The necessity arguments above were proved for Feichtinger-algebra windows, while the existence construction gives Schwartz windows. Hence the same if-and-only-if criterion holds with S replaced by S_0 . The formulas for the minimum number of windows are the least integers satisfying the two branches of mSG_q . $\square$

The proof shows explicitly how the single-window argument is adjusted: replace the N Zibulski–Zeevi columns by qN columns and make no other change in the symplectically rational obstruction. All geometric constructions and all detailed Zak computations in the body of the paper remain single-window arguments.

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Part A is the culmination of a long-running project on the sufficiency of the sharp condition $n \geq m + d$. Before the present argument was found, the authors had obtained partial sufficiency results and a complete proof of the necessity of this condition. A principal motivation for the project was the construction of compactly supported smooth Gabor windows for separable time–frequency lattices.

At an earlier stage, repeated discussions with ChatGPT 5.5 Pro did not resolve either the smallest outstanding example

$$T = 7\mathbb{Z} \times \mathbb{Z}, \quad P = 3\mathbb{Z} \times 3\mathbb{Z},$$

or the general sufficiency problem. A later discussion with ChatGPT 5.6 Pro led to the key idea of replacing hard configurations by soft configurations and studying the resulting extension problem. This soft-configuration viewpoint was the breakthrough that led to the proof of Part A.

After that point, the authors used ChatGPT as an exploratory and drafting tool in the further development, checking, reorganization, and presentation of the remaining arguments, including parts of the Gabor-frame analysis. The authors determined the mathematical objectives and proof strategy, directed the system toward the intended statements, checked the computations and references, and revised the resulting text. The authors take full responsibility for every statement and proof in the paper; no artificial-intelligence system is an author.

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APPENDIX A. A FIRST-PRINCIPLES PFAFFIAN PROOF OF THE COMMON-ZERO THEOREM

This appendix proves the common-zero implication in Theorem 11.3 without vector bundles, Chern classes, differential forms, Sard's theorem, regular-value theory, tangent spaces, orientations, or zero-set manifolds. The argument is entirely coordinate based. Its ingredients are a covariant smoothing operator, ordinary partial derivatives, a real skew-symmetric matrix, the Pfaffian written as an explicit finite sum, finite-dimensional linear algebra, and repeated applications of the fundamental theorem of calculus on a cube.

A continuous function

$$s : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{C}$$

is called *Zak-quasiperiodic* if

$$s(u + n, v + m) = e^{2\pi i n \cdot v} s(u, v), \quad u, v \in \mathbb{R}^d, \quad n, m \in \mathbb{Z}^d. \quad (\text{A.1})$$

A vector-valued function $S = (s_1, \dots, s_N) : \mathbb{R}^{2d} \rightarrow \mathbb{C}^N$ is Zak-quasiperiodic when

$$S(u + n, v + m) = e^{2\pi i n \cdot v} S(u, v). \quad (\text{A.2})$$

Theorem A.1. *Let $1 \leq N \leq d$, and let $s_1, \dots, s_N : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{C}$ be continuous Zak-quasiperiodic functions. Then the functions have a common zero. Equivalently, the map*

$$S = (s_1, \dots, s_N) : \mathbb{R}^{2d} \longrightarrow \mathbb{C}^N$$

cannot be everywhere nonzero.

The proof has four steps.

- (1) A continuous nowhere-zero quasiperiodic map can be smoothed without changing its quasiperiodicity or creating a zero.
- (2) After normalizing a smooth map to have length one, we form scalar coefficients a_r from its first derivatives and a skew-symmetric matrix $F = (\partial_r a_s - \partial_s a_r)$.
- (3) A pointwise rank calculation forces $\text{Pf } F = 0$ everywhere.
- (4) The boundary increments imposed by Zak quasiperiodicity force $\int_{[0,1]^{2N}} \text{Pf } F = 1$.

The last two conclusions contradict one another.

It suffices first to prove the result when $d = N$. Indeed, if $d > N$, we fix the final $d - N$ coordinate pairs and apply the $d = N$ case to the remaining variables. We therefore work from Section A.2 onward with N coordinate pairs.

A.1. Covariant smoothing of continuous quasiperiodic maps

We record the smoothing step because the Pfaffian calculation uses ordinary partial derivatives.

Choose a nonnegative function $\rho \in C_c^\infty(\mathbb{R}^{2N})$ with

$$\int_{\mathbb{R}^{2N}} \rho(x, y) dx dy = 1,$$

and put

$$\rho_\varepsilon(x, y) = \varepsilon^{-2N} \rho(x/\varepsilon, y/\varepsilon).$$

For a continuous Zak-quasiperiodic map $S : \mathbb{R}^{2N} \rightarrow \mathbb{C}^N$, define

$$(\mathcal{M}_\varepsilon S)(u, v) = \int_{\mathbb{R}^{2N}} \rho_\varepsilon(x, y) e^{2\pi i u \cdot y} S(u - x, v - y) dx dy. \quad (\text{A.3})$$

Lemma A.2. *The function $\mathcal{M}_\varepsilon S$ is smooth and satisfies the same Zak boundary rule as S . Moreover,*

$$\mathcal{M}_\varepsilon S \longrightarrow S$$

uniformly on $[0, 1]^{2N}$ as $\varepsilon \rightarrow 0$. Consequently, if S is nowhere zero, then $\mathcal{M}_\varepsilon S$ is nowhere zero for all sufficiently small $\varepsilon > 0$.

Proof. Let $n, m \in \mathbb{Z}^N$. Using quasiperiodicity of S ,

$$\begin{aligned} (\mathcal{M}_\varepsilon S)(u + n, v + m) &= \int \rho_\varepsilon(x, y) e^{2\pi i (u+n) \cdot y} S(u + n - x, v + m - y) dx dy \\ &= \int \rho_\varepsilon(x, y) e^{2\pi i u \cdot y} e^{2\pi i n \cdot y} e^{2\pi i n \cdot (v-y)} S(u - x, v - y) dx dy \\ &= e^{2\pi i n \cdot v} (\mathcal{M}_\varepsilon S)(u, v). \end{aligned}$$

Thus the boundary rule is preserved exactly.

To see smoothness without differentiating the original continuous function, make the change of variables

$$p = u - x, \quad q = v - y.$$

Then

$$(\mathcal{M}_\varepsilon S)(u, v) = \int_{\mathbb{R}^{2N}} \rho_\varepsilon(u - p, v - q) e^{2\pi i u \cdot (v-q)} S(p, q) dp dq. \quad (\text{A.4})$$

For (u, v) in a compact set, the factor $\rho_\varepsilon(u - p, v - q)$ restricts (p, q) to another compact set. Every derivative in u and v falls on the smooth kernel

$$\rho_\varepsilon(u - p, v - q) e^{2\pi i u \cdot (v-q)},$$

so differentiation under the integral is legitimate to every order.

For uniform convergence, rewrite (A.3) as

$$(\mathcal{M}_\varepsilon S)(u, v) - S(u, v) = \int \rho_\varepsilon(x, y) (e^{2\pi i u \cdot y} S(u - x, v - y) - S(u, v)) dx dy.$$

On a slightly enlarged compact fundamental cube, S is uniformly continuous and bounded. The expression in parentheses tends uniformly to zero as $(x, y) \rightarrow (0, 0)$. The usual approximate-identity estimate therefore gives uniform convergence on $[0, 1]^{2N}$.

Finally, $\|S(u, v)\|_2$ is $\mathbb{Z}^{2N}$ -periodic. If S has no zero, then

$$\delta = \min_{[0, 1]^{2N}} \|S(u, v)\|_2 > 0.$$

For sufficiently small ε the uniform approximation error is less than $\delta/2$, and then $\|\mathcal{M}_\varepsilon S\|_2 \geq \delta/2$ everywhere. $\square$

A.2. Normalization and the derivative matrix

Assume for contradiction that a smooth Zak-quasiperiodic map

$$S : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{C}^N$$

is nowhere zero. Normalize it by

$$U(x, y) = \frac{S(x, y)}{\|S(x, y)\|_2}. \quad (\text{A.5})$$

Then

$$U(x, y)^* U(x, y) = 1 \quad (\text{A.6})$$

and

$$U(x + n, y + m) = e^{2\pi i n \cdot y} U(x, y). \quad (\text{A.7})$$

We order the real coordinates as

$$t_1 = x_1, \quad t_2 = y_1, \quad t_3 = x_2, \quad t_4 = y_2, \quad \dots, \quad t_{2N-1} = x_N, \quad t_{2N} = y_N, \quad (\text{A.8})$$

and write $\partial_r = \partial/\partial t_r$.

For $r = 1, \dots, 2N$, define

$$a_r(x, y) = \frac{1}{2\pi i} U(x, y)^* \partial_r U(x, y). \quad (\text{A.9})$$

These coefficients are real-valued. Indeed, differentiating (A.6) gives

$$(\partial_r U)^* U + U^* \partial_r U = 0,$$

so

$$\overline{U^* \partial_r U} = (\partial_r U)^* U = -U^* \partial_r U.$$

Thus $U^* \partial_r U$ is purely imaginary.

For clarity, write

$$a_{x_k} = \frac{1}{2\pi i} U^* \partial_{x_k} U, \quad a_{y_k} = \frac{1}{2\pi i} U^* \partial_{y_k} U.$$

Boundary relations for the coefficients

From

$$U(x + e_j, y) = e^{2\pi i y_j} U(x, y)$$

we obtain

$$a_{x_k}(x + e_j, y) = a_{x_k}(x, y). \quad (\text{A.10})$$

For the y_k derivative,

$$\begin{aligned} \partial_{y_k} U(x + e_j, y) &= \partial_{y_k} \left(e^{2\pi i y_j} U(x, y) \right) \\ &= e^{2\pi i y_j} \left(2\pi i \delta_{jk} U(x, y) + \partial_{y_k} U(x, y) \right). \end{aligned}$$

Multiplying on the left by $U(x + e_j, y)^* = e^{-2\pi i y_j} U(x, y)^*$ gives

$$a_{y_k}(x + e_j, y) = a_{y_k}(x, y) + \delta_{jk}. \quad (\text{A.11})$$

Since U is periodic in every y_j variable,

$$a_r(x, y + e_j) = a_r(x, y). \quad (\text{A.12})$$

Define the real skew-symmetric matrix

$$F = (F_{rs})_{r,s=1}^{2N}, \quad F_{rs} = \partial_r a_s - \partial_s a_r. \quad (\text{A.13})$$

The additive constants in (A.11) disappear after differentiation, so every entry F_{rs} is $\mathbb{Z}^{2N}$ -periodic.

A.3. The Pfaffian and pointwise degeneracy

For a real skew-symmetric $2N \times 2N$ matrix $B = (B_{rs})$, define

$$\text{Pf } B = \frac{1}{2^N N!} \sum_{i_1, \dots, i_{2N}=1}^{2N} \varepsilon_{i_1 \dots i_{2N}} B_{i_1 i_2} B_{i_3 i_4} \cdots B_{i_{2N-1} i_{2N}}, \quad (\text{A.14})$$

where $\varepsilon_{i_1 \dots i_{2N}}$ is the alternating symbol. We use the elementary algebraic identity

$$\det B = (\text{Pf } B)^2. \quad (\text{A.15})$$

It follows directly by expanding both sides as alternating polynomials in the entries of B ; it is also proved by induction using the expansion of the Pfaffian along its first row.

Proposition A.3. *For the matrix F in (A.13),*

$$\text{Pf } F(x, y) = 0$$

at every point (x, y) .

Proof. Differentiate (A.9). The product rule gives

$$\partial_r (U^* \partial_s U) = (\partial_r U)^* \partial_s U + U^* \partial_r \partial_s U,$$

and the same formula with r and s interchanged. Since U is smooth, $\partial_r \partial_s U = \partial_s \partial_r U$, and the second-derivative terms cancel. Hence

$$\begin{aligned} F_{rs} &= \frac{1}{2\pi i} [\partial_r (U^* \partial_s U) - \partial_s (U^* \partial_r U)] \\ &= \frac{1}{2\pi i} [(\partial_r U)^* \partial_s U - (\partial_s U)^* \partial_r U]. \end{aligned} \quad (\text{A.16})$$

Fix a point (x, y) for the remainder of the argument and abbreviate $U = U(x, y)$. Put

$$Q = I_N - UU^*. \quad (\text{A.17})$$

We verify explicitly that Q is the orthogonal projection onto the complex orthogonal complement of U . Since $U^* U = 1$,

$$Q^* = I_N - (UU^*)^* = Q$$

and

$$Q^2 = (I_N - UU^*)^2 = I_N - 2UU^* + U(U^*U)U^* = I_N - UU^* = Q.$$

Moreover,

$$QU = U - U(U^*U) = 0, \quad U^*Q = U^* - (U^*U)U^* = 0.$$

Thus $\text{ran } Q \subseteq U^\perp$. Conversely, if $z \in U^\perp$, then $U^*z = 0$ and therefore $Qz = z$. It follows that

$$\text{ran } Q = U^\perp = \{z \in \mathbb{C}^N : U^*z = 0\}.$$

Because $U \neq 0$, this space has complex dimension $N - 1$ and real dimension $2N - 2$.

For each $r = 1, \dots, 2N$, set

$$w_r = Q \partial_r U, \quad \alpha_r = U^* \partial_r U. \quad (\text{A.18})$$

The vector w_r belongs to $U^\perp$, because $U^*Q = 0$. The identity $I_N = UU^* + Q$ gives the orthogonal decomposition

$$\partial_r U = (UU^* + Q) \partial_r U = U(U^* \partial_r U) + Q \partial_r U = U \alpha_r + w_r.$$

Differentiating $U^*U = 1$ in the t_r direction gives

$$(\partial_r U)^* U + U^* \partial_r U = 0,$$

so $\overline{\alpha_r} = -\alpha_r$. Thus there is a real number β_r with $\alpha_r = i\beta_r$. In particular,

$$\overline{\alpha_r} \alpha_s = (-i\beta_r)(i\beta_s) = \beta_r \beta_s = \overline{\alpha_s} \alpha_r.$$

Also $U^* w_s = 0$ and $w_r^* U = 0$. Consequently,

$$(\partial_r U)^* \partial_s U = (\overline{\alpha_r} U^* + w_r^*)(U \alpha_s + w_s) = \overline{\alpha_r} \alpha_s + w_r^* w_s,$$

$$(\partial_s U)^* \partial_r U = \overline{\alpha_s} \alpha_r + w_s^* w_r.$$

The scalar terms are equal and cancel in (A.16). We therefore obtain

$$F_{rs} = \frac{1}{2\pi i} (w_r^* w_s - w_s^* w_r) = \frac{1}{\pi} \operatorname{Im}(w_r^* w_s). \quad (\text{A.19})$$

The second equality follows from $w_s^* w_r = \overline{w_r^* w_s}$.

We now turn (A.19) into an explicit real matrix factorization. Choose a complex orthonormal basis $e_1, \dots, e_{N-1}$ of $U^\perp$. Write

$$w_r = \sum_{\ell=1}^{N-1} (p_{\ell r} + i q_{\ell r}) e_\ell, \quad p_{\ell r}, q_{\ell r} \in \mathbb{R}.$$

Let W be the real $(2N-2) \times 2N$ matrix whose r th column is

$$W_{\cdot r} = (p_{1r}, q_{1r}, p_{2r}, q_{2r}, \dots, p_{N-1,r}, q_{N-1,r})^T,$$

and let

$$\Omega = \frac{1}{\pi} \operatorname{diag} \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \in M_{2N-2}(\mathbb{R}).$$

A direct calculation gives

$$(W_{\cdot r})^T \Omega W_{\cdot s} = \frac{1}{\pi} \sum_{\ell=1}^{N-1} (p_{\ell r} q_{\ell s} - q_{\ell r} p_{\ell s}) = \frac{1}{\pi} \operatorname{Im}(w_r^* w_s) = F_{rs}.$$

Thus, entry by entry,

$$F = W^T \Omega W. \quad (\text{A.20})$$

When $N = 1$, the space $U^\perp$ is zero-dimensional, W has no rows, and this formula simply says $F = 0$.

Finally,

$$\operatorname{rank} F = \operatorname{rank}(W^T \Omega W) \leq \operatorname{rank} W \leq 2N - 2 < 2N.$$

Hence the $2N \times 2N$ matrix F is singular, and so $\det F = 0$. The identity (A.15) now yields $(\operatorname{Pf} F)^2 = 0$, and therefore $\operatorname{Pf} F = 0$ at the chosen point. Since the point was arbitrary, the conclusion holds everywhere. $\square$

A.4. The boundary-Pfaffian identity and the common-zero contradiction

We now isolate the boundary identity and complete the contradiction argument.

A.4.1. An abstract boundary-Pfaffian identity

The next lemma is independent of the map U . It uses only smooth real functions with the boundary increments found in (A.10)–(A.12). We keep the coordinate order

$$x_1, y_1, x_2, y_2, \dots, x_N, y_N.$$

Thus, inside an alternating symbol, the notation x_j means the index $2j-1$ and the notation y_j means the index $2j$.

Lemma A.4. *For $j = 1, \dots, N$, let a_{x_j}, a_{y_j} be smooth real functions on $\mathbb{R}^{2N}$ satisfying, for every $j, k = 1, \dots, N$,*

$$a_{x_k}(x + e_j, y) = a_{x_k}(x, y), \quad (\text{A.21})$$

$$a_{y_k}(x + e_j, y) = a_{y_k}(x, y) + \delta_{jk}, \quad (\text{A.22})$$

$$a_r(x, y + e_j) = a_r(x, y). \quad (\text{A.23})$$

Order the coordinates as in (A.8), and define $F_{rs} = \partial_r a_s - \partial_s a_r$. Then

$$\int_{[0,1]^{2N}} \operatorname{Pf} F dt_1 \cdots dt_{2N} = 1. \quad (\text{A.24})$$

Before proving the lemma, note that every entry of F is periodic in all $2N$ variables. Indeed, across an x_j -face each coefficient a_s changes by a constant, namely either 0 or 1. Differentiating the boundary relation therefore gives

$$\partial_r a_s(x + e_j, y) = \partial_r a_s(x, y).$$

Across a y_j -face the coefficients themselves are periodic. Hence F_{rs} has identical values on every pair of opposite faces of the unit cube.

The proof of the lemma is a repeated boundary calculation. We first derive an explicit divergence identity.

The divergence identity. For $r = 1, \dots, 2N$, define

$$V_r = \frac{1}{2^{N-1}N!} \sum_{i_2, \dots, i_{2N}=1}^{2N} \varepsilon_{ri_2 \dots i_{2N}} a_{i_2} F_{i_3 i_4} F_{i_5 i_6} \cdots F_{i_{2N-1} i_{2N}}. \quad (\text{A.25})$$

Here and below a term is automatically zero when two indices coincide, because the alternating symbol then vanishes.

Proposition A.5. *The functions in (A.25) satisfy*

$$\sum_{r=1}^{2N} \partial_r V_r = \text{Pf } F. \quad (\text{A.26})$$

Proof. Replace the outer index r by i_1 and apply the product rule. There are two kinds of terms: the derivative may fall on a_{i_2} , or it may fall on one of the $N-1$ factors $F_{i_3 i_4}, \dots, F_{i_{2N-1} i_{2N}}$. Thus

$$\sum_{r=1}^{2N} \partial_r V_r = A + \sum_{p=2}^N B_p,$$

where

$$A = \frac{1}{2^{N-1}N!} \sum_{i_1, \dots, i_{2N}} \varepsilon_{i_1 i_2 \dots i_{2N}} (\partial_{i_1} a_{i_2}) F_{i_3 i_4} \cdots F_{i_{2N-1} i_{2N}}, \quad (\text{A.27})$$

and, for $p = 2, \dots, N$,

$$B_p = \frac{1}{2^{N-1}N!} \sum_{i_1, \dots, i_{2N}} \varepsilon_{i_1 i_2 \dots i_{2N}} a_{i_2} (\partial_{i_1} F_{i_{2p-1} i_{2p}}) \\ \times \prod_{\substack{q=2 \\ q \neq p}}^N F_{i_{2q-1} i_{2q}}.$$

We first compute A . Let

$$\Sigma = \sum_{i_1, \dots, i_{2N}} \varepsilon_{i_1 i_2 \dots i_{2N}} (\partial_{i_1} a_{i_2}) F_{i_3 i_4} \cdots F_{i_{2N-1} i_{2N}}.$$

Interchange the two dummy indices i_1 and i_2 . The alternating symbol changes sign, while the remaining factors are unchanged. Hence the same sum also equals

$$- \sum_{i_1, \dots, i_{2N}} \varepsilon_{i_1 i_2 \dots i_{2N}} (\partial_{i_2} a_{i_1}) F_{i_3 i_4} \cdots F_{i_{2N-1} i_{2N}}.$$

Adding these two representations of Σ gives

$$2\Sigma = \sum_{i_1, \dots, i_{2N}} \varepsilon_{i_1 i_2 \dots i_{2N}} (\partial_{i_1} a_{i_2} - \partial_{i_2} a_{i_1}) F_{i_3 i_4} \cdots F_{i_{2N-1} i_{2N}} \\ = \sum_{i_1, \dots, i_{2N}} \varepsilon_{i_1 i_2 \dots i_{2N}} F_{i_1 i_2} F_{i_3 i_4} \cdots F_{i_{2N-1} i_{2N}}.$$

Therefore the coefficient in (A.27) gives

$$A = \frac{1}{2^N N!} \sum_{i_1, \dots, i_{2N}} \varepsilon_{i_1 \dots i_{2N}} F_{i_1 i_2} \cdots F_{i_{2N-1} i_{2N}} = \text{Pf } F.$$

It remains to prove $B_p = 0$ for each p . The elementary identity used for this cancellation is

$$\partial_r F_{ab} + \partial_a F_{br} + \partial_b F_{ra} = 0. \quad (\text{A.28})$$

To verify it from first principles, substitute $F_{ab} = \partial_a a_b - \partial_b a_a$:

$$\begin{aligned} \partial_r F_{ab} + \partial_a F_{br} + \partial_b F_{ra} \\ = \partial_r \partial_a a_b - \partial_r \partial_b a_a + \partial_a \partial_b a_r - \partial_a \partial_r a_b + \partial_b \partial_r a_a - \partial_b \partial_a a_r = 0. \end{aligned}$$

The first and fourth terms cancel, the second and fifth terms cancel, and the third and sixth terms cancel, because mixed partial derivatives commute.

Fix p and abbreviate $a = i_{2p-1}$ and $b = i_{2p}$. In the sum defining B_p , cyclically rename the three dummy indices

$$(i_1, a, b) \mapsto (a, b, i_1).$$

A three-cycle is an even permutation, so the alternating symbol is unchanged. The value of the sum is therefore unchanged if $\partial_{i_1} F_{ab}$ is replaced by $\partial_a F_{bi_1}$, and it is also unchanged if it is replaced by $\partial_b F_{i_1 a}$. Averaging these three equal representations yields

$$\begin{aligned} B_p &= \frac{1}{3 \cdot 2^{N-1} N!} \sum_{i_1, \dots, i_{2N}} \varepsilon_{i_1 i_2 \dots i_{2N}} a_{i_2} \\ &\quad \times \left(\partial_{i_1} F_{ab} + \partial_a F_{bi_1} + \partial_b F_{i_1 a} \right) \prod_{\substack{q=2 \\ q \neq p}}^N F_{i_{2q-1} i_{2q}} = 0 \end{aligned}$$

by (A.28). Thus every B_p vanishes, while $A = \text{Pf } F$, which proves (A.26). $\square$

Proof of the boundary identity. We now verify the boundary identity by expanding the divergence formula.

Proof of Lemma A.4. We argue by induction on N . Throughout, an integral over a face means integration with respect to all coordinates that remain free on that face.

For $N = 1$, the coordinate order is (x_1, y_1) and

$$\text{Pf } F = F_{x_1 y_1} = \partial_{x_1} a_{y_1} - \partial_{y_1} a_{x_1}.$$

Applying the one-variable fundamental theorem of calculus first in x_1 and then in y_1 gives

$$\begin{aligned} \int_0^1 \int_0^1 \text{Pf } F \, dx_1 \, dy_1 &= \int_0^1 \left(a_{y_1}(1, y_1) - a_{y_1}(0, y_1) \right) dy_1 \\ &\quad - \int_0^1 \left(a_{x_1}(x_1, 1) - a_{x_1}(x_1, 0) \right) dx_1. \end{aligned}$$

The first bracket equals 1 by (A.22), and the second bracket equals 0 by (A.23). Hence the integral equals 1.

Assume now that the assertion has been proved for $N - 1$ coordinate pairs. Integrate (A.26) over the unit cube. For a coordinate t_r , write $dt_{\widehat{r}}$ for integration in all variables except t_r . The fundamental theorem of calculus gives

$$\int_{[0,1]^{2N}} \text{Pf } F = \sum_{r=1}^{2N} \int_{[0,1]^{2N-1}} \left(V_r|_{t_r=1} - V_r|_{t_r=0} \right) dt_{\widehat{r}}. \quad (\text{A.29})$$

We first examine a pair of y_j -faces. By (A.23), every coefficient a_s has the same value at $y_j = 0$ and $y_j = 1$. As observed before the divergence calculation, every entry of F is also periodic. Since V_{y_j} is a polynomial expression in the a_s and F_{rs} , it follows pointwise that

$$V_{y_j}|_{y_j=1} = V_{y_j}|_{y_j=0}.$$

Thus every pair of y_j -faces contributes zero to (A.29).

Now fix j and consider the two x_j -faces. In the notation of (A.25), take $r = x_j$. All F -factors agree on the two faces. For the remaining factor, the boundary relations say

$$a_{i_2}(x + e_j, y) - a_{i_2}(x, y) = \begin{cases} 1, & i_2 = y_j, \\ 0, & i_2 \neq y_j. \end{cases}$$

Therefore only the terms with $i_2 = y_j$ survive in the difference, and

$$\begin{aligned} & V_{x_j}|_{x_j=1} - V_{x_j}|_{x_j=0} \\ &= \frac{1}{2^{N-1}N!} \sum_{i_3, \dots, i_{2N}} \varepsilon_{x_j y_j i_3 \dots i_{2N}} F_{i_3 i_4} \cdots F_{i_{2N-1} i_{2N}}. \end{aligned} \quad (\text{A.30})$$

A nonzero term in this sum must use each of the remaining $2N - 2$ indices exactly once.

Let

$$I_j = \{x_1, y_1, \dots, x_N, y_N\} \setminus \{x_j, y_j\},$$

and let $F^{(j)}$ be the submatrix of F with rows and columns indexed by I_j , evaluated on the face $x_j = 0$. The same submatrix is obtained on $x_j = 1$ because F is periodic. To compare alternating symbols, move the ordered pair (x_j, y_j) from its positions $(2j-1, 2j)$ to the first two positions. Moving x_j requires $2j-2$ transpositions, and after that moving y_j requires another $2j-2$ transpositions. The total number $4j-4$ is even, so no sign is introduced. Hence the alternating symbol in (A.30) is exactly the alternating symbol associated with the inherited order on I_j .

The Pfaffian of the $(2N-2) \times (2N-2)$ matrix $F^{(j)}$ is

$$\text{Pf } F^{(j)} = \frac{1}{2^{N-1}(N-1)!} \sum_{i_3, \dots, i_{2N} \in I_j} \varepsilon_{i_3 \dots i_{2N}}^{(j)} F_{i_3 i_4} \cdots F_{i_{2N-1} i_{2N}},$$

where $\varepsilon^{(j)}$ is the alternating symbol for the inherited coordinate order. Since $N! = N(N-1)!$, comparison with (A.30) gives

$$V_{x_j}|_{x_j=1} - V_{x_j}|_{x_j=0} = \frac{1}{N} \text{Pf } F^{(j)}. \quad (\text{A.31})$$

It remains to integrate this minor. Fix $y_j \in [0, 1]$ and set $x_j = 0$. For every $k \neq j$, restrict the functions a_{x_k} and a_{y_k} to this slice and regard them as functions of the remaining $2N-2$ variables. For shifts in a remaining x_ℓ variable, $\ell \neq j$, they satisfy

$$a_{x_k}(x + e_\ell, y) = a_{x_k}(x, y), \quad a_{y_k}(x + e_\ell, y) = a_{y_k}(x, y) + \delta_{\ell k},$$

and they are periodic in every remaining y_ℓ variable. Thus these restricted functions satisfy exactly the hypotheses of the lemma with $N-1$ coordinate pairs. Their skew derivative matrix, taken only with respect to the remaining variables, is precisely $F^{(j)}$. The induction hypothesis therefore gives

$$\int_{[0,1]^{2N-2}} \text{Pf } F^{(j)} = 1 \quad (\text{A.32})$$

for every fixed value of y_j .

Using (A.31), the total contribution of the two x_j -faces to (A.29) is therefore

$$\frac{1}{N} \int_0^1 \left(\int_{[0,1]^{2N-2}} \text{Pf } F^{(j)} \right) dy_j = \frac{1}{N} \int_0^1 1 dy_j = \frac{1}{N}.$$

There are N choices of j . All y_j -face contributions vanish, and each pair of x_j -faces contributes $1/N$. Hence

$$\int_{[0,1]^{2N}} \text{Pf } F = \sum_{j=1}^N \frac{1}{N} = 1,$$

which completes the induction.

For example, when $N = 2$ and the coordinates are (x_1, y_1, x_2, y_2) ,

$$\text{Pf } F = F_{x_1 y_1} F_{x_2 y_2} - F_{x_1 x_2} F_{y_1 y_2} + F_{x_1 y_2} F_{y_1 x_2}.$$

The argument above says that the y_1 - and y_2 -faces cancel, while the x_1 -faces and the x_2 -faces each contribute $1/2$. $\square$

A.4.2. The contradiction and the continuous theorem

The remaining step rules out a smooth nowhere-zero quasiperiodic map and then returns to the original continuous function by approximation.

Proposition A.6. *There is no smooth nowhere-zero Zak-quasiperiodic map*

$$S : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{C}^N.$$

Proof. Assume that such an S exists. Normalize it to U as in (A.5), and define the coefficients a_r and matrix F as in (A.9) and (A.13). Proposition A.3 gives

$$\text{Pf } F(x, y) = 0$$

for every (x, y) . Therefore

$$\int_{[0,1]^{2N}} \text{Pf } F = 0.$$

On the other hand, the boundary relations (A.10)–(A.12) satisfy the hypotheses of Lemma A.4, which gives

$$\int_{[0,1]^{2N}} \text{Pf } F = 1.$$

This is impossible. $\square$

Proof of Theorem A.1. First suppose $d = N$ and assume that the continuous map $S = (s_1, \dots, s_N)$ is nowhere zero. Lemma A.2 produces, for sufficiently small $\varepsilon > 0$, a smooth nowhere-zero Zak-quasiperiodic map $\mathcal{M}_\varepsilon S$. This contradicts Proposition A.6.

Now let $d > N$. Fix the final $d - N$ coordinate pairs, for example at zero:

$$u_{N+1} = \dots = u_d = 0, \quad v_{N+1} = \dots = v_d = 0.$$

The restricted functions retain the Zak boundary rule in the first N coordinate pairs. The already proved case $d = N$ gives a common zero of the restricted functions, and therefore a common zero of the original functions. $\square$

A.4.3. The first two dimensions

The formulas become especially transparent for small N .

For $N = 1$,

$$F = \begin{pmatrix} 0 & F_{x_1 y_1} \\ -F_{x_1 y_1} & 0 \end{pmatrix}, \quad \text{Pf } F = F_{x_1 y_1}.$$

The boundary calculation is simply

$$\int_{[0,1]^2} (\partial_{x_1} a_{y_1} - \partial_{y_1} a_{x_1}) = 1,$$

which is the ordinary winding-number increment.

For $N = 2$, in the coordinate order (x_1, y_1, x_2, y_2) ,

$$\text{Pf } F = F_{x_1 y_1} F_{x_2 y_2} - F_{x_1 x_2} F_{y_1 y_2} + F_{x_1 y_2} F_{y_1 x_2}.$$

The pointwise rank estimate gives $\text{rank } F \leq 2$, so this expression is zero at every point. The boundary calculation nevertheless gives

$$\int_{[0,1]^4} \text{Pf } F = 1.$$

Thus the higher-dimensional contradiction is an iterated version of the one-dimensional winding computation.

A.4.4. Concluding comment

The proof is entirely coordinate based. Nonvanishing allows us to normalize the tuple. The normalized tuple produces a skew derivative matrix F . Its columns are generated by vectors in a real space of dimension $2N - 2$, so its top Pfaffian vanishes pointwise. The Zak boundary phase, however, contributes one unit of winding in each coordinate pair, and the explicit divergence calculation packages those increments into

$$\int_{[0,1]^{2N}} \text{Pf } F = 1.$$

The contradiction proves the common-zero theorem from ordinary calculus and finite-dimensional linear algebra.

APPENDIX B. A COMPACTLY SUPPORTED PARSEVAL FRAME OUTSIDE THE BLOCK-ZERO CASES

This appendix proves the example stated after Theorem 1.7. The construction is one-dimensional and completely explicit.

Proposition B.1. *Let*

$$a = \frac{71}{100}, \quad \Gamma_0 = a \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \mathbb{Z}^2.$$

There exists a nonzero $g_0 \in C_c^\infty(\mathbb{R})$ such that $\mathcal{G}(g_0, \Gamma_0)$ is an orthonormal system. Consequently, if

$$\Lambda_0 = \Gamma_0^\circ = \frac{1}{2a} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \mathbb{Z}^2,$$

then

$$g = \sqrt{\text{covol}(\Lambda_0)} g_0$$

belongs to $C_c^\infty(\mathbb{R})$ and $\mathcal{G}(g, \Lambda_0)$ is a Parseval frame. Moreover,

$$\Lambda_0 = cS\mathbb{Z}^2, \quad c = \frac{1}{\sqrt{2a}}, \quad S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in \text{Sp}(2, \mathbb{R}),$$

and every entry of S is nonzero.

Proof. Put

$$L = \frac{1}{2a} = \frac{50}{71}.$$

The numerical inequality needed below is

$$L < a, \quad a - L = \frac{41}{7100} > 0.$$

Choose

$$0 < \delta < \frac{a - L}{2};$$

for example, $\delta = (a - L)/4$. Then

$$L + 2\delta < a. \tag{B.1}$$

We first construct a smooth function whose squared modulus has constant L -periodization. Define

$$\rho(s) = \begin{cases} 0, & s \leq 0, \\ e^{-1/s}, & s > 0, \end{cases} \quad \eta(s) = \frac{\rho(s)}{\rho(s) + \rho(1-s)}.$$

Then $\eta \in C^\infty(\mathbb{R})$, $\eta(s) = 0$ for $s \leq 0$, and $\eta(s) = 1$ for $s \geq 1$. Set

$$\vartheta(t) = \frac{\pi}{2} \eta\left(\frac{t + \delta}{2\delta}\right).$$

Thus $\vartheta(t) = 0$ for $t \leq -\delta$, $\vartheta(t) = \pi/2$ for $t \geq \delta$, and all derivatives of ϑ vanish at the two transition endpoints. Define

$$g_0(t) = \frac{1}{\sqrt{L}} \begin{cases} \sin \vartheta(t), & -\delta \leq t \leq \delta, \\ 1, & \delta \leq t \leq L - \delta, \\ \cos \vartheta(t - L), & L - \delta \leq t \leq L + \delta, \\ 0, & \text{otherwise.} \end{cases} \quad (\text{B.2})$$

The flatness of the transition function shows that the four pieces and all their derivatives agree at the endpoints. Hence $g_0 \in C_c^\infty(\mathbb{R})$, and

$$\text{supp } g_0 \subset [-\delta, L + \delta].$$

By (B.1), the diameter of this support is strictly smaller than a .

We claim that

$$\sum_{j \in \mathbb{Z}} |g_0(x + jL)|^2 = \frac{1}{L}, \quad x \in \mathbb{R}. \quad (\text{B.3})$$

It is enough to consider $0 \leq x < L$. If $\delta \leq x \leq L - \delta$, only the term $j = 0$ is nonzero, and its squared modulus is $1/L$. If $0 \leq x < \delta$, the only nonzero terms are $j = 0$ and $j = 1$, and

$$|g_0(x)|^2 + |g_0(x + L)|^2 = \frac{1}{L} (\sin^2 \vartheta(x) + \cos^2 \vartheta(x)) = \frac{1}{L}.$$

If $L - \delta < x < L$, the only nonzero terms are $j = 0$ and $j = -1$, and the same identity, with argument $x - L$, applies. This proves (B.3).

Taking the Fourier coefficients of the L -periodic function in (B.3) gives, for every $k \in \mathbb{Z}$,

$$\begin{aligned} \int_{\mathbb{R}} |g_0(t)|^2 e^{-2\pi i k t / L} dt &= \int_0^L \sum_{j \in \mathbb{Z}} |g_0(x + jL)|^2 e^{-2\pi i k x / L} dx \\ &= \frac{1}{L} \int_0^L e^{-2\pi i k x / L} dx = \delta_{k,0}. \end{aligned} \quad (\text{B.4})$$

In particular, $\|g_0\|_2 = 1$.

We now verify orthogonality on Γ_0 . For $m, n \in \mathbb{Z}$,

$$a \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = (a(m+n), a(-m+n)).$$

Writing $p = m + n$ and $q = -m + n$, we obtain

$$\Gamma_0 = \{(ap, aq) : p, q \in \mathbb{Z}, p \equiv q \pmod{2}\}. \quad (\text{B.5})$$

Let $(ap, aq) \in \Gamma_0 \setminus \{0\}$. If $p \neq 0$, then $|ap| \geq a$, while the support diameter of g_0 is smaller than a . Therefore

$$\text{supp } g_0 \cap (\text{supp } g_0 + ap) = \emptyset,$$

and hence

$$\langle g_0, M_{aq} T_{ap} g_0 \rangle = 0.$$

If $p = 0$, the parity condition in (B.5) gives $q = 2k$ for some $k \in \mathbb{Z} \setminus \{0\}$. Since $2a = 1/L$, we have $aq = k/L$, and (B.4) yields

$$\langle g_0, M_{aq}g_0 \rangle = \int_{\mathbb{R}} |g_0(t)|^2 e^{2\pi ikt/L} dt = 0.$$

Thus g_0 is orthogonal to every nontrivial time–frequency translate of itself by a point of Γ_0 . Since $\|g_0\|_2 = 1$ and the inner product of two lattice atoms differs, up to a constant of modulus one, from an inner product with their lattice difference, $\mathcal{G}(g_0, \Gamma_0)$ is an orthonormal system.

Let

$$M_0 = a \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

For the convention used in this paper, the adjoint lattice of $M_0\mathbb{Z}^2$ is $JM_0^{-T}\mathbb{Z}^2$. A direct calculation gives

$$JM_0^{-T}\mathbb{Z}^2 = \frac{1}{2a} \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \mathbb{Z}^2 = \frac{1}{2a} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \mathbb{Z}^2,$$

where the final equality is only a change of the integer lattice basis. This proves the displayed formula for Λ_0 .

Write

$$c = \frac{1}{\sqrt{2a}} = \frac{100}{71\sqrt{2}}, \quad S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Then $\det S = 1$, so $S \in \mathrm{Sp}(2, \mathbb{R})$, and $\Lambda_0 = cS\mathbb{Z}^2$. All four entries, which are the four scalar blocks in dimension one, are nonzero. Furthermore,

$$\mathrm{covol}(\Lambda_0) = c^2 = \frac{1}{2a^2} = \frac{5000}{5041}.$$

Its symplectic Gram matrix is

$$(cS)^T J (cS) = c^2 J = \frac{5000}{5041} J.$$

Since 5000 and 5041 are coprime, an integer coordinate vector $k \in \mathbb{Z}^2$ represents an element of the integral symplectic subgroup exactly when both coordinates of k are divisible by 5041. Hence

$$[\Lambda_0 : (\Lambda_0)_{\mathrm{int}}] = 5041^2, \quad \nu(\Lambda_0) = 5041.$$

The reciprocal symplectic Gram matrix for the adjoint lattice has denominator 5000, and therefore

$$\nu(\Lambda_0^\circ) = 5000.$$

In particular, Λ_0 satisfies (SG) in dimension one.

Finally, put

$$g = \sqrt{\mathrm{covol}(\Lambda_0)} g_0 = c g_0.$$

Then $\mathrm{covol}(\Lambda_0)^{-1/2} g = g_0$, and $\Lambda_0^\circ = \Gamma_0$. The normalized Gabor duality principle, Theorem 11.1, therefore shows that $\mathcal{G}(g, \Lambda_0)$ is a Parseval frame. Since scalar multiplication does not change support or smoothness, $g \in C_c^\infty(\mathbb{R})$. $\square$

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