MULTIPLE LATTICE TILES AND RIESZ BASES OF EXPONENTIALS

MIHAIL N. KOLOUNTZAKIS

Abstract. Suppose $\Omega \subseteq \mathbb{R}^d$ is a bounded and measurable set and $\Lambda \subseteq \mathbb{R}^d$ is a lattice. Suppose also that Ω tiles multiply, at level k, when translated at the locations Λ. This means that the Λ-translates of Ω cover almost every point of $\mathbb{R}^d$ exactly k times. We show here that there is a set of exponentials $\exp(2\pi i t \cdot x)$, $t \in T$, where T is some countable subset of $\mathbb{R}^d$, which forms a Riesz basis of $L^2(\Omega)$. This result was recently proved by Grepstad and Lev under the extra assumption that Ω has boundary of measure 0, using methods from the theory of quasicrystals. Our approach is rather more elementary and is based almost entirely on linear algebra. The set of frequencies T turns out to be a finite union of shifted copies of the dual lattice Λ^*. It can be chosen knowing only Λ and k and is the same for all Ω that tile multiply with Λ.

Keywords: Riesz bases of exponentials; Tiling

Contents

Notation: We write $e(x) = e^{2\pi i x}$. If E is a set then χ_E is its indicator function. If A is a non-singular $d \times d$ matrix and $\Lambda = A\mathbb{Z}^d$ is a lattice in $\mathbb{R}^d$ then $\Lambda^* = A^{-\top}\mathbb{Z}^d$ denotes the dual lattice.

1. Introduction

1.1. **Riesz bases.** In this paper we deal with the question of existence of a Riesz (unconditional) basis of exponentials

$$e_t(x) := e(t \cdot x) = e^{2\pi i t \cdot x}, \quad t \in L,$$

for the space $L^2(\Omega)$, where $\Omega \subseteq \mathbb{R}^d$ is a domain of finite Lebesgue measure and $L \subseteq \mathbb{R}^d$ is a countable set of frequencies. By Riesz basis we mean that every $f \in L^2(\Omega)$ can be written uniquely in the form

$$f(x) = \sum_{t \in L} a_t \cdot e(t \cdot x) \tag{1}$$

with the coefficients $a_t \in \mathbb{C}$ satisfying

$$C_1 \left\|f\right\|_2^2 \le \sum_{t \in L} |a_t|^2 \le C_2 \left\|f\right\|_2^2, \tag{2}$$

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for some positive and finite constants C_1, C_2.

1.2. **Orthogonal bases.** One very special example of a Riesz basis occurs when the exponentials $e(t \cdot x), t \in L$, can be chosen to be orthogonal and complete for $L^2(\Omega)$. One can then choose $a_t = |\Omega|^{-1/2}\langle f, e_t\rangle$ and $C_1 = C_2 = 1/|\Omega|$ for (2) to hold as an equality. For instance, if $\Omega = (0, 1)^d$ is the unit cube in $\mathbb{R}^d$ then one can take $L = \mathbb{Z}^d$ and obtain such an orthogonal basis of exponentials. This case, where an orthogonal basis of exponentials exists, is a very rigid situation though and many "reasonable" domains do not have such a basis (a ball is one example [4, 10], or any other smooth convex body or any non-symmetric convex body [7]).

The problem of which domains admit an orthogonal basis of exponentials has been studied intensively. The so called Fuglede or Spectral Set Conjecture [4] (claiming that for Ω to have such a basis it is necessary and sufficient that it can tile space by translations) was eventually proved to be false in dimension at least 3 [20, 12, 2, 3, 11], in both directions. Yet the conjecture may still be true in several important special cases such as convex bodies [8], and it generated many interesting results even after the disproof of its general validity (a rather dated account may be found in [10]).

It is expected that the existence of a Riesz basis for a domain Ω is a much more general, and perhaps even generic, phenomenon, although proofs of existence of a Riesz basis for specific domains are still rather rare, especially in higher dimension [13, 14, 16]. Also no domain is known not to have a Riesz basis of exponentials [13].

1.3. **Lattice tiles.** One general class of domains for which an orthogonal basis of exponentials is known to exist is the class of *lattice tiles*. A domain $\Omega \in \mathbb{R}^d$ is said to *tile* space when translated at the locations of the lattice L (a discrete additive subgroup of $\mathbb{R}^d$ containing d linearly independent vectors) if

$$\sum_{t \in L} \chi_\Omega(x - t) = 1, \quad \text{for almost all } x \in \mathbb{R}^d. \tag{3}$$

Intuitively this condition means that one can cover $\mathbb{R}^d$ with the L-translates of Ω, with no overlaps, except for a set of measure zero (usually the translates of $\partial\Omega$, for "nice" domains Ω).

It is not hard to see that when Ω has finite and non-zero measure then the set L has density equal to $1/|\Omega|$. If L is a lattice then we call Ω an *almost fundamental domain* of L and $|\Omega| = (\operatorname{dens} L)^{-1}$. A *fundamental domain* of L is any set which contains exactly one element of each coset mod L, for instance a fundamental parallelepiped. There are of course many others, as indicated in Figure 1.

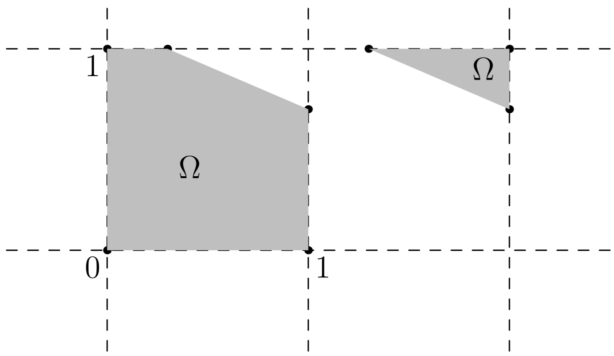

Figure 1: Shaded Ω is a fundamental domain of $\mathbb{R}^2/\mathbb{Z}^2$

It is easy to see [4, 10] that every lattice tile by the lattice L has an orthogonal basis of exponentials, namely those with frequencies $t \in L^*$, where L^* is the dual lattice.

1.4. **Multiple tiling by a lattice.** We say that a domain tiles multiply when its translates cover space the same number of times, almost everyhwere.

Definition 1.1. *Let $\Omega \subseteq \mathbb{R}^d$ be measurable and $L \subseteq \mathbb{R}^d$ be a countable set. We say that Ω tiles $\mathbb{R}^d$ when translated by L at level $k \in \mathbb{N}$ if*

$$\sum_{t \in L} \chi_\Omega(x-t) = k, \tag{4}$$

for almost every $x \in \mathbb{R}^d$. If we do not specify k then we mean $k = 1$.

Multiple tiles are a much wider class of domains that level-one tiles. For instance [1, 9], any centrally symmetric convex polygon in the plane whose vertices have integer coordinates tiles multiply by the lattice $\mathbb{Z}^2$ at some level $k \in \mathbb{N}$. In contrast, only parallelograms or symmetric hexagons can tile at level one.

Another difference is the fact that if two *disjoint* domains Ω_1 and Ω_2 both tile multiply when translated at the locations L then so does their union. In the case of multiple lattice tiling this operation gives essentially the totality of multiple tiles starting from level-one tiles, according to the following easy Lemma.

Lemma 1. *Suppose $\Omega \subseteq \mathbb{R}^d$ is a measurable set which tiles $\mathbb{R}^d$ at level k when translated by the lattice $\Lambda \subseteq \mathbb{R}^d$. Then we can partition*

$$\Omega = \Omega_1 \cup \cdots \cup \Omega_k \cup E, \tag{5}$$

where E has measure 0 and the Ω_j are measurable, mutually disjoint and each Ω_j is an almost fundamental domain of the lattice Λ.

Proof. Let $D \subseteq \mathbb{R}^d$ be a measurable fundamental domain of Λ, for instance one of its fundamental parallelepipeds. For almost every $x \in D$ (call the exceptional set $E \subseteq D$) it follows from our tiling assumption that $\Omega \cap (x + \Lambda)$ contains exactly k points, which we denote by

$$p_1(x) < p_2(x) < \cdots < p_k(x),$$

ordered according to the lexicographical ordering in $\mathbb{R}^d$. We also have that almost every point of Ω belongs to exactly one such list.

Let then $\Omega_j = \bigcup_{x \in D \setminus E} p_j(x)$, for $j = 1, 2, \ldots, k$. In other words, for (almost) each one of the classes mod Λ we distribute its k occurences in Ω into the sets Ω_j. It is easy to see that the Ω_j are disjoint and measurable and that they are almost fundamental domains of Λ. □

1.5. **Multiple lattice tiles have Riesz bases of exponentials.** It is not true that domains that tile multiply by a lattice have an orthogonal basis of exponentials. For instance, it is known [8] that the only convex polygons that have such a basis are parallelograms and symmetric hexagons, yet every symmetric convex polygon with integer vertices is a multiple tile, a much wider class.

It is however true that multiple tiles have a Riesz basis of exponentials. The main result of this paper is the following theorem.

Theorem 1. *Suppose $\Omega \subseteq \mathbb{R}^d$ is bounded, measurable and tiles $\mathbb{R}^d$ multiply at level k with the lattice Λ. Then there are vectors $a_1, \ldots, a_k \in \mathbb{R}^d$ such that the exponentials*

$$e\Big((a_j + \lambda^*) \cdot x\Big), \ j = 1, 2, \ldots, k, \ \lambda^* \in \Lambda^* \tag{6}$$

form a Riesz basis for $L^2(\Omega)$.

The vectors $a_1, \ldots, a_k$ depend on Λ and k only, not on Ω.

Theorem 1 was proved by Grepstad and Lev [6] with the additional topological assumption that the boundary $\partial\Omega$ has Lebesgue measure 0.

In [6] the result is proved following the method of [18, 17] on quasicrystals. Our approach is more elementary and almost entirely based on linear algebra. The authors of [6] have pointed out to me that there are similarities of the method in this paper and the methods in [14, 15, 16]. The method essentially appears also in [19, §3.2].

As an interesting corollary of Theorem 1 let us mention, as is done in [6], that, according to the recent result of [5], if Ω is a centrally symmetric polytope in $\mathbb{R}^d$, whose codimension 1 faces are also centrally symmetric and whose vertices all have rational coordinates, then $L^2(\Omega)$ has a Riesz basis of exponentials.

Open Problem 1. *Is Theorem 1 still true if Ω is of finite measure but unbounded?*

2. Proof of the main result

The essence of the proof is contained in the following lemma.

Lemma 2. *Suppose $\Omega \subseteq \mathbb{R}^d$ is bounded, measurable and tiles $\mathbb{R}^d$ multiply at level k with the lattice Λ. Then there exist vectors $a_1, a_2, \ldots, a_k \in \mathbb{R}^d$ such that the following is true.*

For any $f \in L^2(\Omega)$ there are unique measurable functions $f_j : \mathbb{R}^d \to \mathbb{C}$ such that

1. *The f_j are Λ-periodic,*
2. *The f_j are in L^2 of any almost fundamental domain of Λ, and*
3. *We have the decomposition*

$$f(x) = \sum_{j=1}^{k} e\left(a_j \cdot x\right) f_j(x), \quad \text{for a.e. } x \in \Omega. \tag{7}$$

Finally we have

$$C_1 \left\|f\right\|^2_{L^2(\Omega)} \le \sum_{j=1}^{k} \left\|f_j\right\|^2_{L^2(\Omega)} \le C_2 \left\|f\right\|^2_{L^2(\Omega)}, \tag{8}$$

where $0 < C_1, C_2 < \infty$ do not depend on f.

Proof. Using Lemma 1 we can write Ω as the disjoint union

$$\Omega = \Omega_1 \cup \cdots \cup \Omega_k,$$

where each Ω_k is a measurable almost fundamental domain of Λ. We can now define for $j = 1, 2, \ldots, k$ and for almost every $x \in \mathbb{R}^d$

$$\omega_j(x) \text{ as the unique point in } \Omega_j \text{ s.t. } x - \omega_j(x) \in \Lambda, \text{ and} \tag{9}$$

$$\lambda_j(x) = x - \omega_j(x). \tag{10}$$

(The maps ω_j are clearly measurable and measure-preserving when restricted to a fundamental domain of Λ.) Since the sought-after f_j are to be Λ-periodic it is enough to define them on Ω_1 and extend them to $\mathbb{R}^d$ by their Λ-periodicity. We may therefore rewrite our target decomposition (7) equivalently as follows.

$$\text{For each } x \in \Omega_1 \text{ and } r = 1, 2, \ldots, k\text{: } f(\omega_r(x)) = \textstyle\sum_{j=1}^{k} e\left(a_j \cdot (x - \lambda_r(x))\right) f_j(x). \tag{11}$$

We view (11) as a $k \times k$ linear system

$$M\widetilde{F} = F \tag{12}$$

whose right-hand side is the column vector

$$F = (f(\omega_1(x)), f(\omega_2(x)), \ldots, f(\omega_k(x)))^\top$$

and the unknowns form the column vector

$$\widetilde{F} = (f_1(x), f_2(x), \ldots, f_k(x))^\top.$$

We have a different linear system for each $x \in \Omega_1$ and its matrix is $M = M(x) \in \mathbb{C}^{k\times k}$ with

$$M_{r,j} = M_{r,j}(x) = e\Big(a_j \cdot (x - \lambda_r(x))\Big), \;\; r, j = 1, 2, \ldots, k. \tag{13}$$

Factoring we can write this matrix as

$$M(x) = N(x)\,\mathrm{diag}\,(e\,(a_1 \cdot x), e\,(a_2 \cdot x), \ldots, e\,(a_k \cdot x)), \tag{14}$$

with the matrix $N = N(x)$ given by

$$N_{r,j} = N_{r,j}(x) = e\Big(-a_j \cdot \lambda_r(x)\Big), \;\; r, j = 1, 2, \ldots, k.$$

The key observation here is that when varying $x \in \Omega_1$ the number of different $N(x)$ matrices that arise (the a_j are fixed) is finite and bounded by a quantity that depends on Ω and Λ only. The reason for this is that the vectors $\lambda_r(x)$ are among the Λ vectors in the bounded set $\Omega - \Omega$, hence they take values in a finite set. (This is the only place where the boundedness of Ω is used.)

Let us now see that the vectors $a_1, \ldots, a_k$ can be chosen so that all the (finitely many) possible matrices N are invertible. We have

$$\det N(x) = \sum_{\pi \in S_k} \mathrm{sgn}\,(\pi)\, e\left(-\sum_{j=1}^{k} a_j \cdot \lambda_{\pi_j}(x)\right), \tag{15}$$

where S_k denotes the permutation group on $\{1, 2, \ldots, k\}$. By the definition of the vectors $\lambda_r(x)$ and the disjointness of the sets Ω_r it follows that for each x no two $\lambda_r(x)$ can be the same. View now the expression (15) as a function of the vector $a = (a_1, \ldots, a_k) \in \mathbb{R}^{dk}$. Clearly it is a trigonometric polynomial and it is not identically zero as all the frequencies (for π in the symmetric group S_k)

$$\lambda_\pi(x) = (\lambda_{\pi_1}(x), \ldots, \lambda_{\pi_k}(x)) \in \mathbb{R}^{dk}, \tag{16}$$

are distinct precisely because all the $\lambda_r(x)$ are distinct. Since the zero-set of any trigonometric polynomial (that is not identically zero) is a set of codimension at least 1 it follows that the vectors $a_1, \ldots, a_k$ can be chosen so that all the $N(x)$ matrices that arise are invertible.

Let now $x \in \Omega_1$ and consider the solution of the linear system (12) at x that now takes the form

$$\widetilde{F}(x) = \mathrm{diag}\,(e\,(-a_1 \cdot x), e\,(-a_2 \cdot x), \ldots, e\,(-a_k \cdot x))\, N(x)^{-1} F(x). \tag{17}$$

Since $N(x)$ runs through a finite number of invertible matrices and the diagonal matrix in (17) is an isometry it follows that there are finite constants $A_1, A_2 > 0$, independent of f, such that for any $x \in \Omega_1$ we have

$$A_1 \|F(x)\|_{\ell^2}^2 \le \left\|\widetilde{F}(x)\right\|_{\ell^2}^2 \le A_2 \|F(x)\|_{\ell^2}^2. \tag{18}$$

Integrating (18) over Ω_1 we obtain

$$A_1 \left\|f\right\|_{L^2(\Omega)}^2 \le \sum_{j=1}^{k} \left\|f_j\right\|_{L^2(\Omega_1)}^2 \le A_2 \left\|f\right\|_{L^2(\Omega)}^2. \tag{19}$$

This implies (8) with $C_j = k \cdot A_j$, $j = 1, 2$. To show the uniqueness of the decomposition (7) observe that any such decomposition must satisfy the linear system (17), whose non-singularity has been ensured by our choice of the a_j. □

We can now complete the proof of our main result.

Proof of Theorem 1. Let $f \in L^2(\Omega)$. By Lemma 2 we can write f as in (7). Since the f_j are Λ-periodic and are in L^2 of any almost fundamental domain D of Λ it follows that we can expand each f_j in the frequencies of Λ^* (the dual lattice of Λ)

$$f_j(x) = \sum_{\lambda^* \in \Lambda^*} f_{j,\lambda^*} e\left(\lambda^* \cdot x\right), \ j = 1, 2, \ldots, k, \tag{20}$$

with

$$\left\|f_j\right\|^2_{L^2(D)} = \sum_{\lambda^* \in \Lambda^*} \left|f_{j,\lambda^*}\right|^2, \tag{21}$$

since the exponentials $e(\lambda^* \cdot x)$, $\lambda^* \in \Lambda^*$, form an orthogonal basis of $L^2(D)$ (we assume without loss of generality that $|D| = 1$).

The completeness of (6) follows from (7):

$$f(x) = \sum_{j=1}^{k} \sum_{\lambda^* \in \Lambda^*} f_{j,\lambda^*} e\Big((a_j + \lambda^*) \cdot x\Big). \tag{22}$$

The fact that (6) is a Riesz sequence follows from (8):

$$\frac{k}{C_2} \sum_{j,\lambda^*} \left|f_{j,\lambda^*}\right|^2 \le \left\| \sum_{j,\lambda^*} f_{j,\lambda^*} e\Big((a_j + \lambda^*) \cdot x\Big) \right\|^2_{L^2(\Omega)} \le \frac{k}{C_1} \sum_{j,\lambda^*} \left|f_{j,\lambda^*}\right|^2.$$

As is clear from the proof above, the k-tuples of vectors $a_1, \ldots, a_k$ that appear in Theorem 1 are a generic choice: almost all k-tuples will do. The exceptional set in $\mathbb{R}^{dk}$ is a set of lower dimension.

With a little more care one can see that one can choose the vectors $a_1, \ldots, a_k$ to depend on Λ and k only and not on Ω. In the proof of Lemma 2 the a_j were chosen to ensure that the trigonometric polynomials (15) are all non-zero. Fix Λ and k and form the set of all polynomials of the form (15) which are not identically zero. This set of polynomials is countable and each such polynomial vanishes on a set of codimension at least 1 in $\mathbb{R}^{dk}$. It follows that the union of their zero sets cannot possibly exhaust $\mathbb{R}^{dk}$ and we only have to choose the a_j to avoid that union.

Thus there is a choice of a_j that works for all Ω of the same lattice. This proof does not give uniform values for the constants C_1 and C_2 in (8) though. □

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M.K.: Department of Mathematics, University of Crete, GR-700 13, Heraklion, Crete, Greece
E-mail address: kolount@math.uoc.gr