ON PARTICLES IN EQUILIBRIUM ON THE REAL LINE

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Abstract. We study equilibrium configurations of infinitely many identical particles on the real line or finitely many particles on the circle, such that the (repelling) force they exert on each other depends only on their distance. The main question is whether each equilibrium configuration needs to be an arithmetic progression. Under very broad assumptions on the force we show this for the particles on the circle. In the case of infinitely many particles on the line we show the same result under the assumption that the maximal (or the minimal) gap between successive points is finite (positive) and assumed at some pair of successive points. Under the assumption of analyticity for the force field (e.g., the Coulomb force) we deduce some extra rigidity for the configuration: knowing an equilibrium configuration of points in a half-line determines it throughout. Various properties of the equlibrium configuration are proved.

1. Introduction

In this paper we study configurations of identical particles on the real line, or unit circle, that are in mechanical equilibrium when they exert repelling forces on each other that depend only on their distance. We allow an arbitrary strictly monotone decreasing function $F : \mathbb{R}_+ \to \mathbb{R}_+$ to determine the force between two particles as a function of their distance.

A folklore fact in the study of Wigner crystals is that for an infinite system of particles confined on the real line, or a finite system of particles confined on the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$, and for various natural force fields, the only *ground state*, i.e. the minimiser of the energy of the system, is obtained when the particles are equally spaced[1]. Our first result is that this holds in much greater generality: for every strictly monotone decreasing function $F : \mathbb{R}_+ \to \mathbb{R}_+$, the only configurations of $n \in \mathbf{N}$ particles on $\mathbb{S}^1$ which are in mechanical equilibrium when the force between any two particles at distance d is $F(d)$, are obtained when the distance between any two consequtive particles is constant. By *(mechanical) equilibrium* we mean that the net force tangent to $\mathbb{S}^1$ exerted on each particle is zero (Corollary 3). Similarly, we prove that the only periodic configurations of particles on $\mathbb{R}$ in equilibrium are obtained by equally spacing the particles (Corollary 2). Even more, we prove that any configuration in equilibrium that attains the infimum or supremum of distances between consequtive particles is equally spaced. All these facts follow from a very simple argument (Theorem 1), that, if new, might simplify the proofs of the aforementioned statement about ground states of specific potentials.

If the configuration is allowed to be aperiodic, then the problem is to the best of our knowledge open even for specific force fields like e.g. a Coulomb force $F(d) = d^{-2}$. In fact our original motivation was the following question asked by I. Benjamini [5]

Problem 1. If a configuration of particles on $\mathbb{R}$ is in (mechanical) equilibrium, do all distances between subsequent particles have to be equal?

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[1] https://en.wikipedia.org/wiki/Wigner_crystal

Equilibrium here means that the total force exerted on each particle from each side is finite, and the net force exerted on each particle is zero.

This problem is open for all strictly monotone decreasing functions $F : \mathbb{R}_+ \to \mathbb{R}_+$, and we find it interesting that, although it is not clear that the answer is positive for e.g. $F(d) = d^{-2}$, it is also not clear whether there exists F for which the answer is negative. We prove however that if we nail one of the particles at a fixed position, then we can obtain non-trivial equilibrium configurations for continuous F (Theorem 7).

We also obtain the following result about the Coulomb force (or somewhat more general analytic forces). A configuration in equilibrium with bounded distances between consequtive particles is uniquely determined by any of its *tails* (i.e. co-final subsequences); see Theorem 4.

In the above discussion the particles are tacitly assumed to have equal masses. If we allow them to have different masses, then non-equally-spaced stable configurations do exist as observed by Ulam [4, Chapter VII, §4].

Stable particle configurations for generic force functions are also considered in [2], although with a different focus. For an analogue of Proposition 1 in higher dimensions see [1].

2. No extremal gaps

By an *equilibrium configuration* we mean a bi-infinite sequence of real numbers such that a configuration of particles positioned at those numbers is in equilibrium in the sense defined above. An equilibrium configuration is *trivial*, if it is an arithmetic progression, or in other words, if consequtive particles have equal distances.

For a pair of real numbers x, y, we write xy for the absolute value of the force between a particle at x and a particle at y. By a *gap* we mean the distance (i.e. difference) between two consequtive members of an equilibrium configuration.

Theorem 1. *If an equilibrium configuration has a gap of maximal or minimal length, then it is trivial.*

Proof. Suppose there is a non-trivial equilibrium configuration $\dots, w_2, w_1, x, y, z_1, z_2, \dots$, where the gap $[x, y]$ is maximal (see Fig. 1), i.e. $|x-y| \geq |p-q|$ for any two (consequtive) members p, q of the sequence. Since the equilibrium configuration is not trivial, we may assume without loss of generality that $|x-y| > |x-w_1|$. Writing $F^-(x)$ for the force exerted on a particle at x from the left, we have

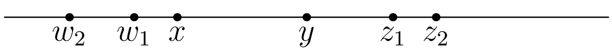

FIGURE 1. The points around a gap of maximum length (x, y).

$$F^-(x) = xw_1 + xw_2 + xw_3 + \dots, \text{and}$$
$$F^-(y) = yx + yw_1 + yw_2 + \dots. \tag{1}$$

Let us compare the jth summand of the first line to the jth summand of the second one: since $|x-y| > |x-w_1|$, we have $yx < xw_1$ by the strict monotonicity of the forces. Moreover, we have $|y-w_i| = |y-x| + |x-w_i| \geq |x-w_{i+1}| = |x-w_i| + |w_i - w_{i+1}|$. Thus $yw_i \leq xw_{i+1}$. Combining these inequalities we obtain $F^-(y) < F^-(x)$.

By repeating the argument for the forces $F^+(y), F^+(x)$ exerted at y, x from the right, the only difference being that $|y-z_1|$ might equal $|x-y|$, we obtain $F^+(y) \geq F^+(x)$, reaching a contradiction.

If the gap $[x, y]$ is minimal, then the same argument applies with all inequalities reversed. $\square$

As an immediate corollary of Theorem 1, we obtain

Corollary 2. *If an equilibrium configuration is periodic, then it is trivial.*

This can be adapted to configurations on the circle S^1:

Corollary 3. *Let $\{x_1, \dots, x_n\}, x_i \in S^1$, be a configuration of particles constrained on S^1 in equilibrium. Suppose that the (tangential) force they exert on each other is a monotone decreasing function of their distance. Then the distance $d(x_i, x_{(i+1) mod n})$ of any two consequtive particles is constant.*

Proof. Suppose, to the contrary, that $d(x_i, x_{(i+1) mod n})$ is not constant. Then there are two consequtive particles x, y maximising that distance, such that the distance between x and its other neighbouring particle w_1 is strictly less that $d(x, y)$. We proceed as in the proof of Theorem 1, the only difference being that now $F^-(x)$ denotes the force exerted on a particle at x by particles lying on one of the two half circles S_x^- between x and its antipodal point x' on S^1.

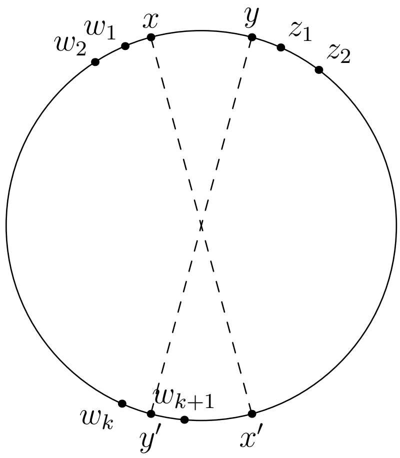

FIGURE 2. The points around an arc of maximum length (x, y).

Thus the sums in (1) have finitely many summands. Since the ith summand of the first sum is greater than the ith summand of the second one by the same argument, it suffices to show that the first sum has at least as many summands as the second. This is indeed true, for if yw_k is the last summand of the second sum, then the particle $w_{(k+1) mod n}$ lies in S_x^- because $d(w_k, w_{(k+1) mod n}) \leq d(x, y)$ by the choice of x, y.

The rest of the proof is identical to that of Theorem 1. $\square$

3. UNIQUENESS OF CONTINUATION UNDER ANALYTIC FORCES

Definition 3.1. *Call an increasing sequence $x_n \in \mathbb{R}$, $n \in \mathbb{Z}$,* uniformly discrete *if there are constants $0 < c \leq C < \infty$ such that*

$$c \leq x_n - x_{n-1} \leq C, \quad \forall n \in \mathbb{Z}.$$

We show the following.

Theorem 4. *Let $x_n \in \mathbb{R}$, $n \in \mathbb{Z}$ be a uniformly discrete configuration of particles subject to repellent Coulomb forces*

$$F(d) = \frac{1}{d^2}.$$

Suppose that the particles at the set $\{x_n \geq 0\}$ are in equilibrium. Then the locations $\{x_n < 0\}$ are uniquely determined.

Proof. Suppose not, and suppose that the two sets of points $X, Y \subseteq (-\infty, 0)$ (each of them uniformly discrete, in the obvious way) can both cause the electrons at the points $W = \{x_n \geq 0\}$ to experience zero total force. In other words, the two systems of electrons, at $X \cup W$ and at $Y \cup W$ are such that the electrons at W are in equilibrium. It follows that for each $w \in W$ the force exerted on w due to electrons at X is the same as the force exerted on w due to electrons at Y.

The Coulomb force exerted at a point w on the nonnegative real semi-axis by the electrons at X is given by

$$f_X(w) = \sum_{x \in X} \frac{1}{(x-w)^2},$$

up to redefining the physical constants, and similarly for the force $f_Y(z)$ due to electrons in Y. Since these must be the same at each $w \in W$ we deduce that the function

$$f(w) = f_X(w) - f_Y(w) = \sum_{p \in X \triangle Y} \frac{\epsilon_p}{(p-w)^2}, \tag{2}$$

where $X \triangle Y$ is the symmetric difference of X and Y and $\epsilon_p = \pm 1$ depending on whether $p \in X$ or $p \in Y$, vanishes at each $w \in W$. It is easy to see that $f(w)$ is well defined (the series at (2) converges) at every point of the complex plane except at $X \triangle Y$, at each point of which it has a pole of order 2, and is an analytic function in $\mathbb{C} \setminus (X \triangle Y)$. Since, for $\Re w \geq 0$ we have

$$\left|f(w)\right| \leq \sum_{p \in X \triangle Y} \frac{1}{\left|p-w\right|^2} \leq \sum_{p \in X \triangle Y} \frac{1}{\left|p\right|^2} < \infty,$$

it is clear that f is bounded on the closed right half plane. Our plan is to use Theorem 5 below to show that f is identically 0.

We write, as we may,

$$W = \{w_0 = 0 < w_1 < w_2, \ldots\}$$

for the points of W and we assume that $c \leq w_n - w_{n-1} \leq C$ for all $n > 0$. This implies that

$$cn \leq w_n \leq Cn, \quad (n \geq 0). \tag{3}$$

Define the linear fractional transformation

$$z = z(w) = \frac{w-1}{w+1}, \quad w = w(z) = \frac{1+z}{1-z}$$

and note that $z(w)$ maps the open right half plane $\{\Re w > 0\}$ bijectively to the open unit disk $\{|z| < 1\}$ (with $1 \to 0, 0 \to -1, i \to i$).

The function $f(w)$ vanishes at all points of W and therefore the analytic function on the unit disk $\{|z| < 1\}$

$$g(z) = f(w(z))$$

vanishes at all (real) points $z_n = z(w_n) = 1 - \frac{2}{w_n+1}$, $n \geq 0$, of the open unit disk. Since f is bounded on the open right half plane so is g on the open unit disk.

Because of (3) we have

$$1 - z_n = \frac{2}{1+w_n} \geq \frac{2}{1+Cn} \tag{4}$$

and hence

$$\sum_n (1 - |z_n|) = \infty. \tag{5}$$

We now use the following result.

Theorem 5 ([3], Theorem 15.23)**.** *If a function g is analytic and bounded in the open unit disk U and vanishes at points $z_n \in U$ satisfying (5), then g is identically 0 in U.*

(This is a rather simple consequence of Jensen's formula.)

Thus Theorem 5 implies that $g \equiv 0$ on U, hence $f \equiv 0$ on the open right half plane, and by analytic continuation f is 0 on $\mathbb{C} \setminus (X \triangle Y)$. So f has no singularities at all, a contradiction, unless $X = Y$, as we had to prove. □

Corollary 6. *Let S be a uniformly discrete equilibrium configuration such that some tail of S is periodic. Then S is trivial.*

Proof. Let T be such a tail, and let T' be the subsequence of T obtained by omitting the first period. By Theorem 4, T can be brought to equilibrium by a unique sequence preceeding it. We claim that this sequence must start with the period of T. Indeed, applying Theorem 4 to T', and noting that T is a shifted copy of T', we see that the two unique continuations coincide.

This easily implies that the whole sequence S is periodic, and by Corollary 2 trivial. □

Generalization. The proof of Theorem 4 and Corollary 6 is valid for more general forces than the Coulomb forces. The force function $F(d)$ needs to be an analytic function on the open right half complex plane, whose values on the positive real axis are positive and satisfies

$$\int_1^{+\infty} F(x)\,dx < \infty.$$

Such functions are, for instance, the functions

$$F(d) = \frac{1}{d^k}, \quad (k \geq 2),$$

and

$$F(d) = e^{-d^k}, \quad (k \geq 1).$$

4. Other remarks

The following facts are easy to check

Proposition 1. *If x, y, z are consequtive points in an equilibrium configuration, then $|x-y|/|y-z|$ is bounded.*

Proposition 1 is an easy consequence of

Proposition 2. *If S is a finite set of consequtive particles in an equilibrium configuration, then the (signed) forces exerted on S by particles in S only are monotone.*

Proof (Sketch). If they are not, then the other forces only make the situation worse. □

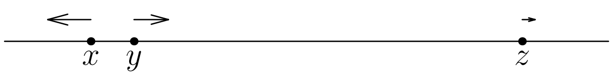

Figure 3. Why Proposition 2 implies Proposition 1.

To see why Proposition 2 implies Proposition 1 let $x < y < z$ be three consecutive points in an equilibrium configuration, hold the points x, y fixed and let z move far to the right (see Fig. 3). Observing the inner forces of the triple we see that if z is far enough to the right then the force on x is negative (it is mostly affected by y), the force on y is positive (it is mostly affected by x) and the force on z is positive but very small, violating the monotonicity proved in Proposition 2.

In fact, the particles in Proposition 2 do not have to be part of an equilibrium configuration; the statement holds for any finite set S of consequtive particles that are in equilibrium inside some configuration. Even stronger, the first particle in S does not have to be in equilibrium.

Proposition 3. *Suppose that the force function F is strictly monotone decreasing and continuous, and $\int_1^\infty F(x)dx < \infty$. For every uniformly discrete sequence of particles $S_- = \{x_{-1}, x_{-2}, \ldots\}$ with $x_i < x_{i-1} < 0$, there is a sequence of particles $S_+ = \{x_0, x_1, x_2, \ldots\}, x_i > x_{i-1} > 0$, such that each particle in S_+ is in equilibrium in the configuration $S_- \cup S_+ = \{x_i\}_{i\in\mathbb{Z}}$. Moreover, x_0 can be chosen arbitrarily.*

Proof. For $n = 1, 2, \ldots$, let x_n be any positive real. Then there are $x_1^n, \ldots x_{n-1}^n \in (0, x_n)$ such that the particles at $\{x_1^n, \ldots x_{n-1}^n\}$ are in equilibrium in the configuration $S_- \cup \{x_1^n, \ldots x_{n-1}^n\} \cup \{x_n\}$: we claim that the positions in $(0, x_n)$ minimising the energy of the particles at $\{x_1^n, \ldots x_{n-1}^n\}$ have this property. To make this argument precise, define the energy $E(x, y)$ contributed by a pair of particles at positions x and y by $E(x, y) := \int_{z=|x-y|}^\infty F(z)dz$. Note that this is finite by the choice of F and the fact that S_- is uniformly discrete.

For $m \in \mathbb{N}_{>0}$, let S_-^m be the subsequence $\{x_{-1}, \ldots, x_{-m}\}$ of S_-. In order to obtain the desired configuration $\{x_1^n, \ldots x_{n-1}^n\}$ we will consider a sequence of configurations $C^m = x_1^{n,m}, \ldots x_{n-1}^{n,m}$, where $x_i^{n,m} \in (0, x_n)$, such that the particles in C^m are in equilibrium in the configuration $S_-^m \cup C^m \cup \{x_n\}$ and use compactness to take a limit.

For this, given m we define the energy $E = E(x_1^n, \ldots x_{n-1}^n)$ of the configuration $S_-^m \cup \{x_1^n, \ldots x_{n-1}^n\} \cup \{x_n\}$ to be

$$E := \sum_{i=0}^{n-1} \sum_{j=-1}^{-m} E(x_i^n, x_j) + \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} E(x_i^n, x_j^n),$$

i.e. the energy contributed by all pairs involving at least one of the particles $\{x_1^n, \ldots x_{n-1}^n\}$. It is not hard to see that there are $x_1^{n,m}, \ldots x_{n-1}^{n,m} \in (0, x_n)$ minimising E by the continuity of F and the fact that E increases if a particle gets too close to x_{-1} or x_n. Note that the partial derivative of E with respect to x_i^n equals the total force exerted on the particle at x_i^n by the definition of E, and by Fermat's theorem this has to vanish at any configuration minimising E. Thus each particle in $\{x_1^{n,m}, \ldots x_{n-1}^{n,m}\}$ is in equilibrium as claimed. By a standard compactness argument, there is a sequence $m_1, m_2, \ldots$ such that the position of x_i^{n,m_j} converges, for each i, as m_j goes to infinity. Define the limit configuration by $x_i^n := \lim x_i^{n,m_j}$. It now follows easily from the continuity of F that the particles at $\{x_1^n, \ldots x_{n-1}^n\}$ are in equilibrium in the configuration $S_- \cup \{x_1^n, \ldots x_{n-1}^n\} \cup \{x_n\}$ as desired.

Moreover, by the monotonicity and continuity of the forces, choosing x_n appropriately we can ensure that x_0 equals any predetermined constant greater than x_{-1}.

By a compactness argument like the one used above, there is a sequence $n_1, n_2, \ldots$ such that the position of $x_i^{n_j}$ converges (possibly to infinity), for each i, as n_j goes to infinity. By Proposition 1 (see also the remark after Proposition 2), the limit of $x_i^{n_j}$ is finite. Then defining $S_+ = \{\lim_j x_i^{n_j}\}_{i\in\mathbb{Z}}$ satisfies our requirements (here, we use the continuity of the forces again). □

Proposition 4. *If the gaps of S_- in Proposition 3 are bounded between real numbers $0 < b < B$, then S_+ can be chosen so that its gaps are bounded between $\min(b, x_0 - x_{-1})$ and $\max(B, x_0 - x_{-1})$.*

Proof. We repeat the proof of Proposition 3, except that we replace the particle at x_n with an 1-way infinite arithmetic progression $x_n, x_n + a, x_n + 2a, \ldots$, where a is any real in (b, B). We claim that, for every $n \in \mathbb{N}$, the resulting gaps of $\{x_1^n, \ldots x_{n-1}^n\}$ are bounded between $\min(b, x_0 - x_{-1})$ and $\max(B, x_0 - x_{-1})$. Indeed, let x, y be the particles spanning the largest (respectively smallest) gap of $\{x_1^n, \ldots x_{n-1}^n\}$. If this gap is longer than $\max(B, x_0 - x_{-1})$ (resp. smaller than $\min(b, x_0 - x_{-1})$), then we can repeat the main argument of the proof of Theorem 1 to obtain a contradiction, since such a gap cannot involve any particle in S_-, and in that proof we only used the equilibrium for the particles x, y. □

We remark that we do not know if Proposition 4 is true for every equilibrium configuration S_+. (We only proved it for one equilibrium configuration.)

Finally, we adapt the proof of Proposition 3 to obtain the main result of this section

Theorem 7. *For every strictly monotone decreasing and continuous force function $F : \mathbb{R}_+ \to \mathbb{R}_+$, there is a configuration $\{\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots\}$ of particles on $\mathbb{R}$ in which all particles except $x_0 = 0$ are in equilibrium, and x_{-1} and x_1 can be chosen arbitrarily.*

Proof. We use the strategy of the proof of Proposition 3, except that we replace the sequence S_- with a single particle at a position x_{-n}, we fix a particle at $x_0 = 0$ which does not have to be in equilibrium, and we introduce particles $\{x^n_{-1}, \dots x^n_{-(n-1)}\}$ in equilibrium for each $n \in \mathbb{N}$. We need to show that, by choosing x_{-n}, x_n appropriately, we can bring the particles x^n_{-1}, x^n_1 to the desired positions for each n. We can then take a limit of such configurations as $n \to \infty$ as in Proposition 3.

Define a *0-centered configuration* to be a sequence $\{x_{-n}, \dots, x_{-1}, x_0 = 0, x_1, \dots x_n\}$ in which all particles except possibly x_{-n}, x_0, x_n are in equilibrium (when forces between particles are given by F). Thus it remains to prove that for every $a < 0, b > 0$, $n \in \mathbb{N}_*$, there is a 0-centered configuration $\{x_{-n}, \dots, x_{-1}, x_0, x_1, \dots x_n\}$ with $x_{-1} = a$ and $x_1 = b$.

To prove this, let

$$d := \sup\{x - x' \mid \text{ there is a 0-centered configuration with } x_{-n} = x', x_n = x, x_{-1} \geq a, \text{ and } x_1 \leq b\}.$$

Let us prove that $d < \infty$. Let $\{x_{-n}, \dots, x_{-1}, x_0, x_1, \dots x_n\}$ be a candidate configuration. Since $x_1 \leq b$, the force to the right exerted on x_1 from particle 0 is at least $F(b)$, and has to be balanced by the particles $x_2, \dots, x_n$. This gives an upper bound b' on x_2, as the force on x_1 to the left is less than $(n-1)F(x_2 - x_1)$ by the monotonicity of F. Similarly, the force to the right exerted on x_2 from particle 0 is lower bounded by $F(b')$, and this imposes an upper bound on x_3, and so on up to x_n. Applying the same argument to the negative particles we also see that x_{-n} is bounded, hence $x_n - x_{-n}$ is bounded.

Since F is continuous, this supremum is attained by some 0-centered configuration $Y = \{y_{-n}, \dots, y_{-1}, y_0, y_1, \dots y_n\}$. We claim that $y_1 = a$ and $y_1 = b$ in C, which would complete our proof.

Suppose to the contrary that $y_{-1} = a + \epsilon$ for some $\epsilon > 0$ (and possibly $y_1 < b$). We will produce a 0-centered configuration $Y' = \{y'_{-n}, \dots, y'_{-1}, y_0 = 0, y'_1, \dots y'_n\}$ where $y'_{-n} = y_{-n} - \epsilon$ and $y'_i \in [y_i - \epsilon, y_i]$ for every i, and in fact $y'_0 = y_0 = 0$ and $y'_n = y_n$. This contradicts the choice of Y as Y' increases d by ϵ, and satisfies all other requirements.

We will obtain Y' as a limit of sequences $Y^j = \{y^j_{-n}, \dots, y^j_{-1}, y_0 = 0, y^j_1, \dots y^j_n\}$, $j = 0, 1, \dots$.

To begin with, we define Y^0 by letting $y^0_{-n} = y_{-n} - \epsilon$, and letting $y^0_i = y_i$ for every other i. In fact, we will never change the position of the particle $-n$ again, that is, we fix $y^j_{-n} = y_{-n} - \epsilon$ for every j. We will also never change the positions of particles 0 and n; we call the particles $-n, 0, n$ the *fixed particles*.

Note that no non-fixed particle is in equilibrium in Y^0: for all non-fixed particles, we have reduced the force from the left in comparison to Y, and kept the force from the right fixed. By the continuity and monotonicity of F, there is a position $y \in (y^0_{-n}, y^0_{-n+1})$ such that if we move the particle $-n+1$ from y^0_{-n+1} to y, then that particle will be in equilibrium. We now define Y^1 by letting $y^1_{-n+1} = y$ and $y^1_i = y^0_i$ for every other i.

We proceed similarly with the next particle $-n+2$. Since we moved the previous two particles to the left, it is still true that the net force on that particle from the left has been reduced, and we move it to the left to a position y^2_{-n+2} to bring it to equilibrium and define Y^2 (we are aware that the particle $-n+1$ is not any more in equilibrium in Y^2).

We proceed inductively to define the sequences $Y^3, Y^4, \dots, Y^{2(n-1)}$, each of which only moves the position of the non-fixed particle $-n+3, \dots, -1, 1, \dots n-1$ respectively to the left. Note that after these changes, all non-fixed particles but $n-1$ are again out of equilibrium, and the net force they experience is to the left. We repeat another round of similar changes, obtaining sequences $Y^{2(n-1)+1}, \dots, Y^{4(n-1)}$ in which particle $-n+1, \dots, -1, 1, \dots n-1$ respectively are moved to the left to reach a temporary equilibrium. After we are done we perform another such round, and so on ad infinitum.

Since each y_i^j is monotone decreasing in j, and bounded below by y_{-n}^0, it converges to some value y_i', and we use these values to define the limit configuration Y'.

Next, we claim that for every j, and every particle i, we have $y_i^j \in [y_i - \epsilon, y_i]$. For if not, then consider the first step j when a counterexample y_i^j arises. Then particle i has to be in equilibrium in Y^j because it must have just been moved. Let us compare the forces exerted on this particle in Y^j to those exerted on it in Y. All particles have been moved to the left if at all, and particle i has experienced the largest displacement as all other particles have moved by at most ϵ. But this means that all particles to the left of i are closer to i in Y^j than they were in Y, and all particles to the right of i are further from i in Y^j than they were in Y. By the strict monotonicity of F, this contradicts the fact that i was in equilibrium in both Y and Y^j.

This proves our claim, which implies that $y_i' \in [y_i - \epsilon, y_i]$ for every i. In particular, $y_{-1}' \geq y_{-1} - \epsilon$, and so $y_{-1}' \geq a$ (and clearly $y_1' \leq b$). Thus Y' is a candidate for the definition of d if it is 0-centered. And indeed it is: since the positions of all particles converge in the sequence (Y^j), the net force on each particle i converges as $j \to \infty$ by the continuity of F, and as it assumes the value 0 infinitely often —namely, at steps at which we bring i to equilibrium— it has to converge to 0.

Thus Y' contradicts the choice of Y as claimed, which proves that $y_{-1} = a$. By the same arguments, we obtain a contradiction if $y_1 < b$ by moving all particles to the right a bit. □

5. Questions

Theorem 4 says that if a 1-way infinite sequence of particles S_+ (at bounded distances) can be brought to equilibrium by another 1-way infinite sequence S_-, then S_+ uniquely determines S_-. We can ask if the converse can be proved: if S_- can be used to bring some S_+ to equilibrium, is S_+ uniquely determined by S_-?

Acknowledgement

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