TILES WITH NO SPECTRA

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Abstract. We exhibit a subset of a finite Abelian group, which tiles the group by translation, and such that its tiling complements do not have a common spectrum (orthogonal basis for their L^2 space consisting of group characters). This disproves the Universal Spectrum Conjecture of Lagarias and Wang [7]. Further, we construct a set in some finite Abelian group, which tiles the group but has no spectrum. We extend this last example to the groups $\mathbb{Z}^d$ and $\mathbb{R}^d$ (for $d \geq 5$) thus disproving one direction of the Spectral Set Conjecture of Fuglede [1]. The other direction was recently disproved by Tao [12].

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1. Introduction

Let G be a locally compact Abelian group and $\Omega \subseteq G$ be a bounded open set. We call Ω *spectral* if there is a set Λ of continuous characters of G which forms an orthogonal basis for $L^2(\Omega)$. Such a set Λ is called a *spectrum* of Ω. This paper concerns a conjecture of Fuglede [1] (the *Spectral Set Conjecture*), which states that a domain Ω in $\mathbb{R}^d$ is spectral if and only if it can tile $\mathbb{R}^d$ by translation. A set Ω tiles $\mathbb{R}^d$ by translation if there exists a set $T \subseteq \mathbb{R}^d$ (called a *tiling complement* of Ω) of translates such that $\sum_{t\in T} \chi_\Omega(x-t) = 1$, for almost all $x \in \mathbb{R}^d$. Here χ_Ω denotes the indicator function of Ω.

Tao [12] has recently proved that the direction "spectral $\Rightarrow$ tiling" does not hold (in dimension 5 and higher – Matolcsi [9] has reduced this dimension to 4). Here we prove that the direction "tiling $\Rightarrow$ spectral" is also false in dimension 5 and higher.

The Spectral Set Conjecture has attracted considerable attention in the last decade, revealing a wealth of connections to functional analysis, combinatorics, commutative algebra, number theory and Fourier analysis (the papers [1, 2, 3, 4, 5, 6, 7, 8, 10, 12] and references therein give a more or less complete account of results related to Fuglede's conjecture). Until Tao's example [12] there had been many results for special cases of domains, tiling complements or spectra, all of them supporting the conjecture. (Already in [1] Fuglede showed that the conjecture is true if either the tiling complement or the spectrum is assumed to be a lattice.) Despite the failure of the conjecture in general, it may still be true for some rather large natural class of domains, such as the convex domains [3].

The counterexample of Tao [12] to the "spectral $\Rightarrow$ tiling" direction was based, originally, on the existence of (real) Hadamard matrices whose size is not a power of 2. Such matrices immediately lead to counterexamples in appropriate finite groups, due to divisibility reasons. The main difficulty in disproving the "tiling $\Rightarrow$ spectral" direction is the lack of natural necessary conditions (which would play the role of divisibility) for a set in order to be spectral.

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In order to produce a counterexample our strategy is as follows. The Spectral Set Conjecture makes sense in finite groups as well, and we first disprove the direction "tiling ⇒ spectral" in an appropriate finite group in §3. This we do by first finding a counterexample to the Universal Spectrum Conjecture of Lagarias and Wang [7]. (This conjecture states, in a finite group, that if a set T can tile the group with tiling complements $T_1, \ldots, T_n$ then these sets are all spectral and share a common spectrum. Note that this conjecture is stronger than the original Spectral Set Conjecture.) In §4, using the example found in the finite group setting, we produce a counterexample in the group $\mathbb{Z}^d$ and finally in $\mathbb{R}^d$, where the Spectral Set Conjecture was originally stated.

In §2 we give necessary background material and describe notation.

2. Preliminaries

Suppose Ω is a bounded open set in a locally compact Abelian group G. We will only be interested in finite groups, $\mathbb{Z}^d$ and $\mathbb{R}^d$ and the forhtcoming considerations apply to them.

We call Ω *spectral* if $L^2(\Omega)$ has an orthogonal basis

$$\Lambda \subseteq \widehat{G}$$

of characters ($\widehat{G}$ denotes the dual group of G [11]). The set Λ is then called a *spectrum* for Ω, and (Ω, Λ) is called a spectral pair in G. In the groups we are dealing with the characters are functions of the type $x \to \exp(2\pi i \langle \nu, x \rangle)$, where ν takes values in an appropriate subgroup of the torus $\mathbb{T}^d$ (if G is discrete) or in Euclidean space.

The inner product and norm on $L^2(\Omega)$ are

$$\langle f, g \rangle_\Omega = \int_\Omega f\overline{g}, \quad \text{and} \quad \|f\|_\Omega^2 = \int_\Omega |f|^2.$$

If $\lambda, \nu \in \widehat{G}$ we have

$$\langle \lambda, \nu \rangle_\Omega = \widehat{\chi_\Omega}(\nu - \lambda).$$

which gives

$$\Lambda \text{ is an orthogonal set} \Leftrightarrow \forall \lambda, \mu \in \Lambda, \lambda \neq \mu : \ \widehat{\chi_\Omega}(\lambda - \mu) = 0$$

For Λ to be complete as well we must in addition have (Parseval)

$$\forall f \in L^2(\Omega) : \quad \|f\|_2^2 = \frac{1}{|\Omega|} \sum_{\lambda \in \Lambda} |\langle f, \lambda \rangle|^2. \tag{1}$$

For the groups we care about (finite groups, $\mathbb{Z}^d$ and $\mathbb{R}^d$) in order for Λ to be complete it is sufficient to have (1) for any character $f \in \widehat{G}$, since then we have it in the closed linear span of these functions, which is all of $L^2(\Omega)$. An equivalent reformulation for Λ to be a spectrum of Ω is therefore that

$$\sum_{\lambda \in \Lambda} |\widehat{\chi_\Omega}|^2(x - \lambda) = |\Omega|^2, \tag{2}$$

for almost every $x \in \widehat{G}$. For finite sets Ω (the group is finite or $\mathbb{Z}^d$) for a set $\Lambda \subseteq \widehat{G}$ to be a spectrum it is necessary and sufficient that Λ satisfy the two conditions:

(a) $\Lambda - \Lambda \subseteq \{\widehat{\chi_\Omega} = 0\} \cup \{0\}$ (orthogonality), and
(b) $\#\Lambda = \#\Omega$ (maximal dimension).

For subsets $\Omega \subseteq \mathbb{R}^d$, when the spectra are infinite, we fall back on (2).

If $f \geq 0$ is in $L^1(G)$ and $T \subseteq G$ we say that f tiles with T at level ℓ if $\sum_{t\in T} f(x-t) = \ell$ for almost all $x \in G$. We denote this by "$f + T = \ell G$" and we call T a *tiling complement* of f. If $f = \chi_\Omega$ is the indicator function of some set then we just write $\Omega + T = \ell G$ instead of $\chi_\Omega + T = \ell G$, and, in this case, if ℓ is not specified it assumed to be 1.

In the finite group case it is immediate to show that $f + T$ is a tiling of G if and only if

$$\left\{\widehat{f} = 0\right\} \cup \{\widehat{\chi_T} = 0\} \cup \{0\} = \widehat{G}. \tag{3}$$

There are analogs of this relationship that hold in the infinite case as well but we will not need these here (see [5]).

If f is a continuous function we write $Z(f)$ for its zero set. For a set A we write Z_A for the zero set of the Fourier Transform of its indicator function $Z(\widehat{\chi_A})$.

The starting point of our considerations is a generalization of the composition construction appearing in [9], Proposition 2.1.

Proposition 2.1. *Let G be a finite Abelian group, and $H \leq G$ a subgroup. Let $T_1, T_2, \dots T_k \subset H$ be subsets of H such that they share a common tiling set in H; i.e. there exists a set $T' \subset H$ such that $T_j + T' = H$ is a tiling for all $1 \leq j \leq k$. Consider any tiling decomposition $S + S' = G/H$ of the factor group G/H, with $\#S = k$, and take arbitrary representatives $s_1, s_2, \dots s_k$ from the cosets of H corresponding to the set S. Then the set $\Gamma := \cup_{j=1}^k (s_j + T_j)$ is a tile in the group G.*

Proof. The proof is simply the observation that for any system of representatives $\tilde{S}' := \{s'_1, s'_2, \dots\}$ of S' the set $T' + \tilde{S}'$ is a tiling set for Γ in G. □

Despite the proof being obvious, this construction seems to include a large class of tilings and it leads to some interesting examples.

When taking $T_1 = T_2 = \cdots = T_k = T$ we (essentially) get back the 'tiling part' of the statement of Proposition 2.1 in [9]. The drawback of that statement, in producing a counterexample to the Spectral Set Conjecture, is that the same construction applies to spectral sets as well (see the 'spectral part' of Proposition 2.1 in [9]). The essence of the generalization here comes from allowing *differrent* sets $T_1, \dots T_k$ to be used.

Let us see the analogous construction for spectral sets.

Proposition 2.2. *Let G be a finite Abelian group, and $H \leq G$ a subgroup. Let $T_1, T_2, \dots T_k \subset H$ be subsets of H such that they share a common spectrum in $\widehat{H}$; i.e. there exists a set $L \subset \widehat{H}$ such that L is a spectrum of T_j for all $1 \leq j \leq k$. Consider any spectral pair (Q, Q') in the factor group G/H, with $\#Q = k$, and take arbitrary representatives $q_1, q_2, \dots q_k$ from the cosets of H corresponding to the set Q. Then the set $\Gamma := \cup_{j=1}^k (q_j + T_j)$ is spectral in the group G.*

Proof. We do not give a detailed proof of this statement, as we will not directly use it in the forthcoming arguments. Let us give an outline of the proof only. For any $l_j \in L$ there exists a $g_j \in \widehat{G}$ such that $g_j|_H = l_j$. Take such characters $g_1, \dots, g_r$ (where $r = \#T_j$). Also, the characters of G which take constant values in cosets of H can be identified with elements of $\widehat{G/H}$. Take such characters $v_1, v_2, \dots, v_k$ corresponding to the elements of Q'. Then the spectrum of Γ is the set $\Lambda = g_j v_l$ $(1 \leq j \leq r$,$1 \leq l \leq k)$. The calculations proving orthogonality proceed along the same line as in [9], Proposition 2.1. Completeness then follows from the cardinality of Λ.

□

The main point of the two preceding constructions is that they are not entirely "compatible". That is, one can hope to find sets $T_1, \dots T_k \subset H$ sharing a common tiling complement T' but not sharing a common spectrum L. This would be a counterexample to the Universal Spectrum Conjecture. Then the construction of Proposition 2.1 will lead to a set Γ which tiles G, but there is nothing to guarantee that Γ is spectral in G (in fact, we will find a way to guarantee that Γ is *not* spectral). This is exactly the route we will follow in §3.

3. Counterexamples in finite groups

Here we follow the path outlined in §2 in order to produce an example of a set Γ in a finite group G, such that Γ is a tile but is not spectral in G.

The first step is to find a counterexample to the Universal Spectrum Conjecture. We are looking for a finite group G and a tile T' in G such that the tiling complements $T_1, \dots T_k$ of T' do not posess a common spectrum L.

For a given $T' \subset G$, one sufficient condition for the existence of a universal spectrum L, as pointed out in [8], is to ensure that

$$\#L \cdot \#T' = \#G \quad \text{and} \quad L - L \subset Z^c_{T'}. \tag{4}$$

Indeed, any tiling complement T_j of T' must satisfy $Z_{T_j} \supset Z^c_{T'} \setminus \{0\}$, therefore condition (4) ensures that L is a spectrum of T_j. (We do not know whether condition (4) is also necessary for the existence of a universal spectrum, as suggested in [8] in the remarks following Theorem 3.1.) Therefore, when trying to construct a set T' having no universal spectrum, one must exclude the existence of a set L satisfying (4).

Notice, that if L satisfies (4) then L is not only a universal spectrum for all tiling complements of T', but also a universal tiling set of all spectra of T'. Indeed, for any spectrum Q of T' we have $Q - Q \subset Z_{T'} \cup \{0\}$ therefore $(L - L) \cap (Q - Q) = \{0\}$ and $\#L \cdot \#Q = \#\widehat{G}$, which ensures that $L + Q = \widehat{G}$ is a tiling.

Having this observation in mind, one way to exclude the possibility of (4) is to choose a set T' which posseses a particular spectrum Q which does not tile $\widehat{G}$ (but recall that T' itself must tile G otherwise the notion of universal spectrum is meaningless). In other words, in some group $\widehat{G}$ take a spectral set Q which does not tile $\widehat{G}$ (such examples already exist, cf. [12], [9]) and choose any spectrum of Q as a candidate for $T' \subset G$. However, the examples in [12] and [9] are such that $\#Q$ does not divide $\#\widehat{G}$, therefore any choice for T' is also doomed by divisibility reasons, because T' cannot tile G either. We circumvent this problem by increasing the size of the group G.

The ideas above are summarized in the following theorem, which disproves the Universal Spectrum Conjecture. (In what follows, the notation $\mathbb{Z}_n$ refers to the cyclic group $\mathbb{Z}/n\mathbb{Z}$.)

Theorem 3.1. *Consider $G = \mathbb{Z}_6^5$ and $E = \{\mathbf{0}, \boldsymbol{e_1}, \boldsymbol{e_2}, \dots \boldsymbol{e_5}\}$ where $\boldsymbol{e_j} = (0, \dots 1, \dots, 0)^\top$. The set E tiles G but has no universal spectrum in $\widehat{G}$.*

Proof. We identify the elements of G and $\widehat{G}$ with column and row vectors, respectively.

The existence of a universal spectrum L is equivalent to the conditions $\#L = 6^4$ and $L - L \subset (\bigcap_j Z_{T_j}) \cup \{0\}$, where T_j are all the tiling complements of E.

Consider the set $K \subset \widehat{G}$ (cf. [9, Theorem 3.1]) consisting of the rows of the following matrix

$$K := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 4 & 4 \\ 2 & 0 & 4 & 4 & 2 \\ 2 & 4 & 0 & 2 & 4 \\ 4 & 4 & 2 & 0 & 2 \\ 4 & 2 & 4 & 2 & 0 \end{pmatrix} \tag{5}$$

(We remark, that K is a spectrum of E. In fact, this follows from the fact that the matrix

$$K' := \frac{1}{3}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 2 & 2 & 1 \\ 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 2 & 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 2 & 1 & 0 \end{pmatrix} \tag{6}$$

is log-Hadamard (i.e. the matrix $U_{jk} = \exp(2\pi i K'_{jk})$ is orthogonal.). We will not use the fact that K is a spectrum, but it reflects the considerations preceding the theorem.)

Observe that K is contained in the subgroup $H \leq \widehat{G}$ of row-vectors having even coordinates only. However, $\#H = 3^5$ and $\#K = 6$, therefore K cannot tile H and, consequently, it cannot tile $\widehat{G}$ either. (It is easy to see that if a set tiles a group then it tiles the subgroup it generates.) It is also easy to check that the set $K - K$ consists of $\mathbf{0}$ and only coordinate permutations of the vector $(0, 2, 2, 4, 4)$. (In fact $K - K$ contains *all* coordinate permutations of $(0, 2, 2, 4, 4)$, but we do not need this.)

Next we show that E admits some tiling complements $T_0, \dots T_{14}$, which have no common spectrum.

Take the vector $\boldsymbol{v_0} = (1, 2, 3, 4, 5)^\top$ and define a group homomorphism $\phi : G \to \mathbb{Z}_6$ by

$$\phi(\boldsymbol{x}) := \boldsymbol{v_0}^\top \boldsymbol{x} \pmod 6.$$

Then ϕ is one-to-one on E, and the image of E is the whole group $\mathbb{Z}_6$. Therefore $T_0 = \ker \phi$ is a tiling complement for E. Notice that $Z^c_{T_0}$ contains all multiples of $\boldsymbol{v_0}^\top$, and, in particular, it contains $2\boldsymbol{v_0}^\top = (2, 4, 0, 2, 4)$, and $4\boldsymbol{v_0}^\top = (4, 2, 0, 4, 2)$. By appropriate permutations of the coordinates of $\boldsymbol{v_0}$ we can define vectors $\boldsymbol{v_1}, \dots, \boldsymbol{v_{14}}$ and corresponding tiling sets $T_0, \dots, \mathcal{T}_{14}$ in such a way that $(\bigcap_{j=0}^{14} Z_{T_j}) \cap (K - K) = \{0\}$. Therefore, a set L satisfying $\#L = 6^4$ and $L - L \subset (\bigcap_{j=0}^{14} Z_{T_j}) \cup \{0\}$ cannot exist because in that case $L + K$ would be a tiling of $\widehat{G}$, and we already know that K is not a tile. □

Having found a counterexample to the Universal Spectrum Conjecture, we use the construction of §2 to exhibit the failure of the Spectral Set Conjecture in finite groups.

Theorem 3.2. *Consider $G_2 = \mathbb{Z}_6^5 \times \mathbb{Z}_{15}$ and $\Gamma = \bigcup_{j=0}^{14} (f_j + \tilde{T}_j)$, where $f_j = (0, 0, 0, 0, 0, j)^\top$ and $\tilde{T}_j$ are the sets appearing in Theorem 3.1 extended by 0 as the last coordinate. Then Γ is a tile in G_2 but it is not spectral.*

Proof. In this proof the notation $\tilde{A}$ always refers to a set $A \subset G = \mathbb{Z}_6^5$ (or, $A \subset \widehat{G} = \mathbb{Z}_6^5$ as row vectors) extended by 0 as the last coordinate.

The fact that Γ is a tile follows from Proposition 2.1 or can easily be seen directly: the tiling complement of Γ is $\tilde{E}$.

To see that Γ is not spectral, note first that the set $\tilde{K}$ is contained in the subgroup $\tilde{H}$ (K and H are defined in the proof of Theorem 3.1), therefore it cannot tile $\widehat{G_2}$ because of divisibility reasons.

Any spectrum Q of Γ must satisfy $\#Q = \#\Gamma = 6^4 \cdot 15$ and $Q - Q \subset Z_\Gamma \cup \{0\}$. Consider the vector $\tilde{\boldsymbol{k_1}} = (0, 2, 2, 4, 4, 0) \in \tilde{K} - \tilde{K}$. We show that $\tilde{\boldsymbol{k_1}} \notin Z_\Gamma$. Indeed,

$$\widehat{\chi}_\Gamma(\tilde{\boldsymbol{k_1}}) = \sum_{j=0}^{14} \widehat{\chi}_{T_j}(\boldsymbol{k_1}) > 0$$

because each term is nonnegative (each T_j being a subgroup in G), and at least one term is non-zero by the construction of Theorem 3.1. The same argument shows that $\tilde{\boldsymbol{k_j}} \notin Z_\Gamma$ for all $\boldsymbol{k_j} \in K - K$.

Therefore, any spectrum Q of Γ must satisfy $(Q - Q) \cap (\tilde{K} - \tilde{K}) = \{0\}$, which, since $\#Q \cdot \#\tilde{K} = \#\widehat{G_2}$, implies $Q + \tilde{K} = \widehat{G_2}$. This is a contradiction since $\tilde{K}$ is not a tile in $\widehat{G_2}$ $\square$

4. Transition to $\mathbb{Z}^d$ and $\mathbb{R}^d$

We now describe a general transition scheme from the finite group setting to $\mathbb{Z}^d$ and $\mathbb{R}^d$. As a result we find a set in $\mathbb{R}^6$, which is a finite union of unit cubes (placed at points with integer coordinates), which tiles $\mathbb{R}^6$ by translations but is not spectral.

First we prove this in the group $\mathbb{Z}^d$.

Theorem 4.1. *Let $\boldsymbol{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ and consider a set $A \subset G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_d}$. Write*

$$T = T(\boldsymbol{n}, k) = \{0, n_1, 2n_1, \ldots, (k-1)n_1\} \times \cdots \times \{0, n_d, 2n_d, \ldots, (k-1)n_d\}, \tag{7}$$

and define $A(k) = A + T$. Then, for large enough values of k, the set $A(k) \subset \mathbb{Z}^d$ is spectral in $\mathbb{Z}^d$ if and only if A is spectral in G.

Proof. The 'if' part of the theorem is essentially contained in [9, Proposition 2.1] (but we will not need this direction here).

To see the 'only if' part, observe first that $\chi_{A(k)} = \chi_A * \chi_T$, hence, writing $Z(f) = \{f = 0\}$, we obtain

$$Z(\widehat{\chi_{A(k)}}) = Z(\widehat{\chi_A}) \cup Z(\widehat{\chi_T}).$$

Elementary calculation of $\widehat{\chi_T}$ (it is a cartesian product) shows that it is a union of "hyperplanes"

$$Z(\widehat{\chi_T}) = \left\{ \boldsymbol{\xi} \in \mathbb{T}^d : \ \exists j \ \exists \nu \in \mathbb{Z}, \ k \text{ does not divide } \nu, \text{ such that } \xi_j = \frac{\nu}{kn_j} \right\}. \tag{8}$$

Define the group

$$H = \left\{ \boldsymbol{\xi} \in \mathbb{T}^d : \ \forall j \ \exists \nu \in \mathbb{Z} \text{ such that } \xi_j = \frac{\nu}{n_j} \right\}.$$

which is the group of characters of the group G and does not depend on k. Observe that $H + (Q - Q)$ does not intersect $Z(\widehat{\chi_T})$, where

$$Q = \left[0, \frac{1}{kn_1}\right) \times \cdots \times \left[0, \frac{1}{kn_d}\right).$$

Assume now that $S \subseteq \mathbb{T}^d$ is a spectrum of $A(k)$, so that $\#S = \#A(k) = rk^d$, if $r = \#A$. Define, for $\boldsymbol{\nu} \in \{0, \ldots, k-1\}^d$, the sets

$$S_{\boldsymbol{\nu}} = \left\{ \boldsymbol{\xi} \in S : \ \boldsymbol{\xi} \in (\frac{\nu_1}{kn_1}, \ldots, \frac{\nu_d}{kn_d}) + Q + (\frac{m_1}{n_1}, \ldots, \frac{m_d}{n_d}), \text{ for some } \boldsymbol{m} \in \mathbb{Z}^d \right\}.$$

Since the number of the $S_{\boldsymbol{\nu}}$ is k^d and they partition S, it follows that there exists some $\boldsymbol{\mu}$ for which $\#S_{\boldsymbol{\mu}} \ge r$.

We also note that, if k is sufficiently large, then any translate of Q may contain at most one point of the spectrum. The reason is that $Q - Q$ contains no point of $Z(\widehat{\chi_T})$ (for any k) and no point of $Z(\widehat{\chi_A})$ for all large k (as $\widehat{\chi_A}(\boldsymbol{0}) > 0$).

Observe next that if $\boldsymbol{x}, \boldsymbol{y} \in S_{\boldsymbol{\mu}}$ then

$$\begin{aligned} \boldsymbol{x} - \boldsymbol{y} &\in H + (Q - Q) \\ &= H + \left(-\frac{1}{kn_1}, \frac{1}{kn_1}\right) \times \cdots \times \left(-\frac{1}{kn_d}, \frac{1}{kn_d}\right) \end{aligned}$$

and that this set does not intersect $Z(\widehat{\chi_T})$ (from (8)). It follows that for all $\boldsymbol{x}, \boldsymbol{y} \in S_{\boldsymbol{\mu}}$ we have $\boldsymbol{x} - \boldsymbol{y} \in Z(\widehat{\chi_A})$.

Let k be sufficiently large so that for all points $\boldsymbol{h} \in H$ for which $\widehat{\chi_A}(\boldsymbol{h}) \neq 0$ the rectangle $\boldsymbol{h} + Q - Q$ does not intersect $Z(\widehat{\chi_A})$. It follows that if $\boldsymbol{x}, \boldsymbol{y} \in S_{\boldsymbol{\mu}}$ then $\boldsymbol{x} - \boldsymbol{y} \in \boldsymbol{h} + (Q - Q)$, where $\boldsymbol{h} \in Z(\widehat{\chi_A})$.

For each $\boldsymbol{x} \in \mathbb{T}^d$ define $\lambda(\boldsymbol{x})$ to be the unique point $\boldsymbol{z}$ whose j-th coordinate is an integer multiple of $\frac{1}{kn_j}$ for which $\boldsymbol{x} \in \boldsymbol{z} + Q$. If $\boldsymbol{x}, \boldsymbol{y} \in S_{\boldsymbol{\mu}}$ it follows that $\lambda(\boldsymbol{x}) - \lambda(\boldsymbol{y}) \in H \cap Z(\widehat{\chi_A})$. Define now $\Lambda = \{\lambda(\boldsymbol{x}) : \ \boldsymbol{x} \in S_{\boldsymbol{\mu}}\}$ (and shift Λ to contain 0, so that $\Lambda \subset H$). It is obvious that $\#\Lambda \ge r$ and $\Lambda - \Lambda \subseteq Z(\widehat{\chi_A}) \cup \{\boldsymbol{0}\}$, therefore Λ is a spectrum of A.

□

Non-spectral tiles can be pulled from $\mathbb{Z}^d$ to $\mathbb{R}^d$ using the following.

Theorem 4.2. *Suppose $A \subseteq \mathbb{Z}^d$ is a finite set and $Q = [0,1)^d$. Then A is a spectral set in $\mathbb{Z}^d$ if and only if $A + Q$ is a spectral set in $\mathbb{R}^d$.*

Proof. Write $E = A + Q$. Then $\widehat{\chi_E} = \widehat{\chi_A}\widehat{\chi_Q}$ and $Z(\widehat{\chi_E}) = Z(\widehat{\chi_A}) \cup Z(\widehat{\chi_Q})$. By calculation we have

$$Z(\widehat{\chi_Q}) = \{\boldsymbol{\xi} \in \mathbb{R}^d : \ \exists j \text{ such that } \xi_j \in \mathbb{Z} \setminus \{\boldsymbol{0}\}\}.$$

Now suppose $\Lambda \subset \mathbb{T}^d$ is a spectrum of A as a subset of $\mathbb{Z}^d$. Viewing $\mathbb{T}^d$ as Q we observe that the set $Z(\widehat{\chi_A})$ is periodic with $\mathbb{Z}^d$ as a period lattice. Define now $S = \Lambda + \mathbb{Z}^d$. The differences of S are either points which are on $Z(\widehat{\chi_A})$ (mod $\mathbb{Z}^d$) or points with all integer coordinates. In any case these differences fall in $Z(\widehat{\chi_E})$, hence $\sum_{s \in S} |\widehat{\chi_E}(x - s)|^2 \le (\#A)^2$. Furthermore, the density of S is $\#A$ which, along with the periodicity of S, implies that $|\widehat{\chi_E}|^2 + S$ is a tiling of $\mathbb{R}^d$ at level $(\#A)^2$. That is, S is a spectrum for E.

Conversely, assume S is a spectrum for E as a subset of $\mathbb{R}^d$. It follows that the density of S is equal to $|E| = \#A$, hence there exists $\boldsymbol{k} \in \mathbb{Z}^d$ such that $\boldsymbol{k} + Q$ contains at least $\#A$ points of S. Call the set of these points S_1, and observe that the differences of points of S_1 are contained in $Q - Q = (-1,1)^d$, and that $Q - Q$ does not intersect $Z(\widehat{\chi_Q})$. It follows that the differences of the points of S_1 are all in $Z(\widehat{\chi_A})$, and, since their number is $\#A$, they form a spectrum of A as a subset of $\mathbb{Z}^d$. □

In conclusion we have proved the following.

Theorem 4.3. *In each of the groups $\mathbb{Z}^d$ and $\mathbb{R}^d$, $d \geq 5$, there exists a set which tiles the group by translation but is not spectral*

Proof. It is easy to see that if A is our non-spectral tile in the finite group $\mathbb{Z}_6^5 \times \mathbb{Z}_{15}$ then the set $A(k) \subseteq \mathbb{Z}^6$ which appears in Theorem 4.1 is a tile as well, and by that Theorem it is not spectral. Using Theorem 4.2 we can construct a set with these properties in $\mathbb{R}^6$ by adding a unit cube at each point.

To get down to dimension 5, notice that the construction in Theorem 3.2 can be repeated verbatim in the group $G_3 = \mathbb{Z}_6^5 \times \mathbb{Z}_{17}$ instead of the group $\mathbb{Z}_6^5 \times \mathbb{Z}_{15}$. (Just repeat the set T_{15} two more times.) But now we view G_3 as the group $\mathbb{Z}_6^4 \times \mathbb{Z}_{6\cdot 17}$ (as 6 and 17 are coprime). Theorems 4.1 and 4.2 now give examples in dimension 5.

If $d \geq 6$ then the set of $\mathbb{Z}^6$ which is tiling but not spectral will still be such in $\mathbb{Z}^d$ when viewed as a subset of that in the obvious way. The preservation of tiling property is obvious, and one can easily show that the existence of any spectrum in $\mathbb{T}^d$ implies the existence of a spectrum in $\mathbb{T}^6$. We go to $\mathbb{R}^d$ again using Theorem 4.2. □

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