

FUNCTIONS WITH COMPARABLE INTEGRALS ON ALL k -PLANES

MIHAIL N. KOLOUNTZAKIS AND GÖTZ E. PFANDER

ABSTRACT. Let $d \geq 2$ and $1 \leq k \leq d - 1$. We show that if $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is nonnegative and measurable and $0 < m \leq M < \infty$ then it is impossible that on almost all affine k -planes P in $\mathbb{R}^d$ the integral of f on P lies between m and M . Let $G = \mathbb{Z}^k \times \{0\}^{d-k}$. Using $m = M = 1$ and f being the indicator function of a measurable set in $\mathbb{R}^d$ this then implies that there is no measurable Steinhaus set for the group G in $\mathbb{R}^d$. In other words there is no measurable set $S \subseteq \mathbb{R}^d$ such that S tiles $\mathbb{R}^d$ with $T(G)$, for all $T \in O(d)$. We also generalize our impossibility results concerning the size of line-integrals of functions to integrals on strips in the plane.

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1. INTRODUCTION

A classical question of Steinhaus [Sie58, Mos81] asks if there is a set $S \subseteq \mathbb{R}^2$ such that for all $\theta \in [0, 2\pi)$ S tiles $\mathbb{R}^2$ when translated at the locations $R_\theta \mathbb{Z}^2$, where R_θ denotes rotation by θ around the origin. This means that every $x \in \mathbb{R}^2$ should belong to exactly one translate $S + n$, for $n \in R_\theta \mathbb{Z}^2$. In a major result Jackson and Mauldin [JM02b, JM02a, Mau, JM03] proved the existence of lattice Steinhaus sets in the plane (not necessarily measurable).

We are interested in the version of this problem where, on the one hand, we demand that S is Lebesgue measurable and, on the other, we view the tiling $S + R_\theta \mathbb{Z}^2$ as an almost everywhere property: for all θ , almost every $x \in \mathbb{R}^2$ belongs to exactly one translate $S + n$, where $n \in R_\theta \mathbb{Z}^2$. It is easy to see that if S is a measurable

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Steinhaus set then its measure must be 1. Measurable Steinhaus sets are not known to exist when tiling by $\mathbb{Z}^2$ and its rotates.

Regarding the measurable version of the problem Sierpinski [Sie58] showed that a bounded set which is either closed or open cannot have the Steinhaus property while Croft [Cro82] and Beck [Bec89] showed that this is also impossible for any bounded and measurable set (see also [MR95]). Kolountzakis [Kol96b, Kol96a] and Kolountzakis and Wolff [KW99] proved that any measurable set in the plane that has the measurable Steinhaus property must necessarily have very slow decay at infinity. In [KW99] it was also shown that there can be no measurable Steinhaus sets in dimension $d \geq 3$ (here we are seeking a measurable subset of $\mathbb{R}^d$ which tiles with every rotate $T(\mathbb{Z}^d)$, where $T \in O(d)$). This was reproved later by Kolountzakis and Papadimitrakakis [KP02] by a very different method. See also [CM07, MY03, Ciu96, ST05]. Kolountzakis [Kol97] looks at the case where the linear transformation ρ is only required to take on finitely many values and the question is studied there for when finitely many lattices of the same volume in $\mathbb{R}^d$ can have the same fundamental domain. Especially for the case of two lattices there have been some major results recently [GK26, Spy26].

The Steinhaus tiling problem makes sense for more general subsets of $\mathbb{R}^2$ or $\mathbb{R}^d$. If $\Gamma \subseteq \mathbb{R}^d$ we say that the measurable $S \subseteq \mathbb{R}^d$ has the Steinhaus property with respect to Γ if $S + T(\Gamma)$ is a tiling for all $T \in O(d)$. It was shown in [KP17] that for any finite $\Gamma \subseteq \mathbb{R}^d$ there is no measurable set in $\mathbb{R}^d$ with the Steinhaus property.

In [KP17] it was also shown that if $\Gamma = \mathbb{Z} \times \{0\} \subseteq \mathbb{R}^2$ then there is also no measurable set in $\mathbb{R}^2$ with the Steinhaus property. It is this last result that concerns us in this paper. It was shown in [KP17] that if such a set S existed (in the plane) and $f = \mathbf{1}_S$ then $\int_L f = 1$ on almost all lines L in the plane. This was then proved to be impossible, thus disproving the existence of sets S with the Steinhaus property with respect to $\mathbb{Z} \times \{0\}$ in the plane.

The essential part of the proof of this fact was to prove that it is impossible for a set S in the plane with indicator function f to satisfy

$$(1) \quad m \leq \int_L f \leq M, \quad \text{for almost all lines } L,$$

for some constants $0 < m \leq M < \infty$. (For the application to Steinhaus sets it was enough to have $m = M = 1$.) This was proved using the theory of harmonic functions. The proof in [KP17] introduces, for $S \subset \mathbb{R}^2$, the function in $\mathbb{R}^3$

$$(2) \quad F(z) = \int_{\mathbb{R}^2} \frac{\mathbf{1}_S(w)}{|z - w|} dw, \quad z \in \mathbb{R}^3,$$

shows that its boundary values on $\mathbb{R}^2$ are constant, proves that F is continuous and harmonic in the upper half-space, and then uses the Poisson representation for bounded harmonic functions to obtain a contradiction with the decay of F in the vertical direction.

The main contribution of this paper is to give an alternative, elementary proof of the impossibility of (1), while at the same time generalizing from indicator functions to all nonnegative functions (a modification of the proof in [KP17] can also be made to work for nonnegative functions) and generalizing also from line integrals to integrals over affine k -planes in $\mathbb{R}^d$.

There is some related literature in integral geometry and tomography. For instance Ilmavirta and Paternain [IP19] prove that a smooth bounded strictly convex Euclidean domain admits an L^1 density integrating to one over almost every line meeting the domain if and only if the domain is a ball. Armitage [Arm94] and Zalcman [Zal82] construct signed functions on the plane whose integrals along all lines are 0. What distinguishes our problem is the absence of integrability or support conditions on the function.

Thus our main result, whose proof is presented in Section 2, is:

Theorem 1. *Let $f \geq 0$ be a nonnegative Lebesgue measurable function on $\mathbb{R}^d$, $d \geq 2$, and let $1 \leq k \leq d - 1$. Then there are no constants $0 < m \leq M < \infty$ for which*

$$(3) \quad m \leq \int_P f d\mathcal{H}^k \leq M, \quad \text{for almost all } k\text{-planes } P.$$

In the following proposition we show that if S has the Steinhaus property with respect to $\mathbb{Z}^k \times \{0\}^{d-k}$ in $\mathbb{R}^d$ then its indicator function would have to satisfy (3).

Proposition 1. *Let $1 \leq k < d$ and $G = \mathbb{Z}^k \times \{0\}^{d-k} \subseteq \mathbb{R}^d$, let $S \subset \mathbb{R}^d$ be measurable, and write $f = 1_S$. Assume that for almost every $T \in O(d)$, $\sum_{n \in T(G)} f(x - n) = 1$ for almost every $x \in \mathbb{R}^d$. Then, for almost every $x \in \mathbb{R}^d$, one has $\mathcal{H}^k(S \cap P) = 1$ for almost every affine k -plane P through x .*

Proof. Put $E_0 = \mathbb{R}^k \times \{0\}^{d-k}$. Fix a rotation T for which the hypothesis holds, and set $E = T(E_0)$ and $\Lambda = T(G)$. Writing $\mathbb{R}^d = E^\perp \oplus E$ and applying Fubini, for almost every $y \in E^\perp$ we have $\sum_{n \in \Lambda} f(y + u - n) = 1$ for almost every $u \in E$. Integrating this for u in a fundamental domain Q of Λ in E gives $\mathcal{H}^k(S \cap (y + E)) = 1$. Hence, for almost every T , we have $\mathcal{H}^k(S \cap (x + T(E_0))) = 1$ for almost every $x \in \mathbb{R}^d$. Using Fubini again we obtain that for a.e. x , this equality holds for a.e. $T \in O(d)$, which is the desired conclusion. □

Proposition 1 and Theorem 1 immediately imply

Corollary 1. *If $d \geq 2$ and $1 \leq k \leq d - 1$ then there are no measurable sets with the Steinhaus property with respect to $\mathbb{Z}^k \times \{0\}^{d-k}$.*

If the nonnegative, measurable function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ has line integrals between m and M along almost all straight lines of the plane then for almost all directions u and almost every strip S in the plane, parallel to u , of width R the integral $\int_S f$ lies in

$[mR, MR]$. We know there are no functions with line integrals in $[m, M]$ but are there any functions whose integrals over any strip S of width R satisfy $mR \leq \int_S f \leq MR$? The following corollary shows this is not the case.

Corollary 2. *For any $0 < m \leq M < \infty$ there is no nonnegative, measurable function f on $\mathbb{R}^2$ such that if S is any strip and R is its width then $mR \leq \int_S f \leq MR$.*

Proof. Fix a unit vector u and, for $t \in \mathbb{R}$, let $L_t = \{x : x \cdot u = t\}$. For almost every t put $F_u(t) = \int_{L_t} f$. By Fubini, for every $a \in \mathbb{R}$ and $R > 0$,

$$(4) \quad m \leq \frac{1}{R} \int_a^{a+R} F_u(t) dt \leq M.$$

In particular $F_u \in L^1_{\text{loc}}(\mathbb{R})$. By the Lebesgue differentiation theorem, letting $R \rightarrow 0$ in (4) gives

$$(5) \quad m \leq F_u(t) \leq M$$

for a.e. t . Since this holds for every direction u , it follows that (5) holds for almost every line in the plane, contradicting Theorem 1. □

Key in the proof of Corollary 2 was the fact that we could use strips of arbitrarily small width. It turns out that this is not necessary.

Corollary 3. *For any $0 < m \leq M < \infty$ there is no nonnegative, measurable function f on $\mathbb{R}^2$ such that if S is any strip of width 1 then $m \leq \int_S f \leq M$.*

Proof. Let $\phi(x) = \frac{1}{\pi} e^{-|x|^2}$, $x \in \mathbb{R}^2$, and $g = f * \phi$. The convolution is well defined because of the decay of the gaussian and the assumption on the integrals of f on strips of width 1.

Fix a unit vector u and define the measure μ_u on $\mathbb{R}$ that arises by projecting of f onto $\mathbb{R}u$:

$$\mu_u(I) = \int_{\{x: x \cdot u \in I\}} f(x) dx. \quad (I \subseteq \mathbb{R})$$

For every interval I of length 1 we have then $m \leq \mu_u(I) \leq M$.

The one-dimensional marginal of ϕ is $h(t) = \frac{1}{\sqrt{\pi}} e^{-t^2}$. Consequently the projection of $g(x)$ onto $\mathbb{R}u$ has density

$$(6) \quad q_u(t) = \int_{\mathbb{R}} h(t-s) d\mu_u(s).$$

From (6) we easily obtain constants $m_0, M_0 > 0$, depending only on m and M , such that

$$(7) \quad m_0 \leq q_u(t) \leq M_0 \quad \text{for every } u, t.$$

If S is now any strip of arbitrary width $R > 0$, corresponding under $x \mapsto x \cdot u$ to an interval J of length R , then $\int_S g = \int_J q_u(t) dt$ and (7) gives

$$m_0 R \leq \int_S g \leq M_0 R.$$

This contradicts Corollary 2.

□

2. PROOF OF THEOREM 1

Let $f \geq 0$ satisfy (3). We may assume, translating f , that almost all k -planes P through the origin satisfy $m \leq \int_P f d\mathcal{H}^k \leq M$. Then all rotates $f(T\cdot)$, with $T \in O(d)$, satisfy (3), and so does their average $\int_{O(d)} f(T\cdot) dT$, so we may assume that $f(x) = f(r)$, $r = |x|$, is radial. Integrating f along any non-exceptional k -plane through the origin shows that

$$(8) \quad \int_0^\infty f(r) r^{k-1} dr < \infty.$$

Write $F(t)$ for the integral of f along any k -plane at distance t from the origin (since f is radial this does not depend on the chosen plane, only on t). We obtain

$$(9) \quad F(t) = \sigma_{k-1} \int_t^\infty f(r) r (r^2 - t^2)^{\frac{k-2}{2}} dr$$

and for a.e. t we have $m \leq F(t) \leq M$. (Here σ_ℓ denotes the surface area of a the unit sphere in $\mathbb{R}^{\ell+1}$.)

For $k \geq 2$ we have from (9)

$$F(t) \leq \sigma_{k-1} \int_t^\infty f(r) r^{k-1} dr,$$

hence $\lim_{t \rightarrow \infty} F(t) = 0$ from (8), which contradicts the assumed lower bound m for $F(t)$.

For $k = 1$ equation (9) becomes

$$(10) \quad F(t) = 2 \int_t^\infty f(r) r (r^2 - t^2)^{-\frac{1}{2}} dr.$$

We will show that

$$\lim_{R \rightarrow \infty} \frac{1}{R} \int_R^{2R} F(t) dt = 0,$$

thus arriving again at contradiction with $F(t) \geq m$.

We have (Fubini)

$$(11) \quad \begin{aligned} \frac{1}{R} \int_R^{2R} F(t) dt &= \frac{2}{R} \int_R^\infty f(r) r \left(\int_R^{\min\{2R, r\}} \frac{dt}{\sqrt{r^2 - t^2}} \right) dr \\ &\leq \frac{2}{R} \int_R^\infty f(r) \sqrt{r} \left(\int_R^{\min\{2R, r\}} \frac{dt}{\sqrt{r - t}} \right) dr. \end{aligned}$$

We show that for $r \geq R$

$$(12) \quad S(r, R) := \sqrt{r} \int_R^{\min\{2R, r\}} \frac{dt}{\sqrt{r - t}} = 2 \sqrt{r} (\sqrt{r - R} - \sqrt{r - \min\{2R, r\}}) \leq CR$$

for some constant C .

If $r \geq 2R$ then

$$\begin{aligned} S(r, R) &= 2 \sqrt{r} (\sqrt{r - R} - \sqrt{r - 2R}) \\ &= \frac{2R \sqrt{r}}{\sqrt{r - R} + \sqrt{r - 2R}} \\ &\leq \frac{2R \sqrt{r}}{\sqrt{r/2}} \\ &\leq CR. \end{aligned}$$

If $r \leq 2R$ then

$$S(r, R) = 2 \sqrt{r} \sqrt{r - R} \leq 2 \sqrt{2R} \sqrt{2R - R} \leq CR,$$

which completes the proof of (12).

It follows that

$$\frac{1}{R} \int_R^{2R} F(t) dt \leq C \int_R^\infty f(r) dr$$

which tends to 0 as $R \rightarrow \infty$ from (8), a contradiction with $F(t) \geq m$.

This completes the proof of Theorem 1.

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REFERENCES

- [Arm94] D. H. Armitage. A non-constant continuous function on the plane whose integral on every line is zero. *The American Mathematical Monthly*, 101(9):892–894, 1994.
- [Bec89] J. Beck. On a lattice point problem of H. Steinhaus. *Studia Sci. Math. Hung.*, 24:263–268, 1989.
- [Ciu96] M. Ciucu. A remark on sets having the Steinhaus property. *Combinatorica*, 16(3):321–324, 1996.

- [CM07] W. K. Chan and R. Mauldin. Steinhaus tiling problem and integral quadratic forms. *Proc. Am. Math. Soc.*, 135(2):337–342, 2007.
- [Cro82] H. T. Croft. Three lattice-point problems of Steinhaus. *Q. J. Math.*, 33(1):71–83, 1982.
- [GK26] S. Grepstad and M. N. Kolountzakis. Bounded common fundamental domains for two lattices. *Advances in Mathematics*, 487:110776, 2026.
- [IP19] J. Ilmavirta and G. P. Paternain. Functions of constant geodesic x-ray transform. *Inverse Problems*, 35(6):065002, 2019.
- [JM02a] S. Jackson and R. Mauldin. On a lattice problem of H. Steinhaus. *J. Am. Math. Soc.*, 15(4):817–856, 2002.
- [JM02b] S. Jackson and R. D. Mauldin. Sets meeting isometric copies of the lattice $\mathbb{Z}^2$ in exactly one point. *Proc. Natl. Acad. Sci. USA*, 99(25):15883–15887, 2002.
- [JM03] S. Jackson and R. D. Mauldin. Survey of the Steinhaus tiling problem. *Bull. Symb. Log.*, 9(03):335–361, 2003.
- [Kol96a] M. N. Kolountzakis. A new estimate for a problem of Steinhaus. *Int. Math. Res. Not.*, 1996(11):547–555, 1996.
- [Kol96b] M. N. Kolountzakis. A problem of Steinhaus: Can all placements of a planar set contain exactly one lattice point? *Progress in Mathematics*, 139:559–566, 1996.
- [Kol97] M. N. Kolountzakis. Multi-lattice tiles. *Int. Math. Res. Not.*, 1997(19):937–952, 1997.
- [KP02] M. N. Kolountzakis and M. Papadimitrakis. The Steinhaus tiling problem and the range of certain quadratic forms. *Ill. J. Math.*, 46(3):947–951, 2002.
- [KP17] M. N. Kolountzakis and M. Papadimitrakis. Measurable Steinhaus sets do not exist for finite sets or the integers in the plane. *Bull. Lond. Math. Soc.*, 49(5):798–805, 2017.
- [KW99] M. N. Kolountzakis and T. H. Wolff. On the Steinhaus tiling problem. *Mathematika*, 46(02):253–280, 1999.
- [Mau] R. Mauldin. Some problems in set theory, analysis and geometry, Paul Erdős and his mathematics, I (Budapest, 1999), 493–506. *Bolyai Soc. Math. Stud.*, 11.
- [Mos81] W. Moser. *Research problems in discrete geometry*. Department of Mathematics, McGill University, 1981.
- [MR95] E. Mallinikova and S. Rukshin. On one Steinhaus problem. *Vestn. St. Petersburg Univ., Math.*, 28(1):28–32, 1995.
- [MY03] R. Mauldin and A. Yingst. Comments about the Steinhaus tiling problem. *Proc. Am. Math. Soc.*, 131(7):2071–2079, 2003.
- [Sie58] W. Sierpiński. Sur un probleme de H. Steinhaus concernant les ensembles de points sur le plan. *Fundam. Math.*, 2(46):191–194, 1958.
- [Spy26] E. Spyridakis. Bounded and measurable common fundamental domains for two lattices. *arXiv preprint arXiv:2602.23480*, 2026.
- [ST05] S. M. Srivastava and R. Thangadurai. On Steinhaus sets. *Expo. Math.*, 23(2):171–177, 2005.
- [Zal82] L. Zalcman. Uniqueness and nonuniqueness for the Radon transform. *Bulletin of the London Mathematical Society*, 14(3):241–245, 1982.

DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS, UNIVERSITY OF CRETE,, VOUTES CAMPUS,
70013 HERAKLION, GREECE,

AND

INSTITUTE OF COMPUTER SCIENCE, FOUNDATION OF RESEARCH AND TECHNOLOGY HELLAS, N. PLASTIRA
100, VASSILIKA VOUTON, 700 13, HERAKLION, GREECE

Email address: kolount@uoc.gr

CHAIR OF MATHEMATICS – SCIENTIFIC COMPUTING, CATHOLIC UNIVERSITY OF EICHSTÄTT–INGOLSTADT,
MIDS, AUF DER SCHANZ 49, 85049 INGOLSTADT, GERMANY

Email address: pfander@ku.de