The Fourier Transform and applications

Mihalis Kolountzakis

University of Crete

January 2006
Locally compact abelian groups:

Haar measure on $G = \text{translation invariant on } G$: $\mu(A) = \mu(A + t)$. Unique up to scalar multiple.
Groups and Haar measure

Locally compact abelian groups:
- Integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

Haar measure on $G = \text{translation invariant on } G$: $\mu(A) = \mu(A + t)$. Unique up to scalar multiple.
- Counting measure on $\mathbb{Z}$
Groups and Haar measure

Locally compact abelian groups:

- Integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- Finite cyclic group $\mathbb{Z}_m = \{0, 1, \ldots, m - 1\}$: addition mod $m$

Haar measure on $G = \text{translation invariant on } G$: $\mu(A) = \mu(A + t)$. Unique up to scalar multiple.

- Counting measure on $\mathbb{Z}$
- Counting measure on $\mathbb{Z}_m$, normalized to total measure 1 (usually)
Groups and Haar measure

Locally compact abelian groups:
- Integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- Finite cyclic group $\mathbb{Z}_m = \{0, 1, \ldots, m - 1\}$: addition mod $m$
- Reals $\mathbb{R}$

Haar measure on $G =$ translation invariant on $G$: $\mu(A) = \mu(A + t)$. Unique up to scalar multiple.
- Counting measure on $\mathbb{Z}$
- Counting measure on $\mathbb{Z}_m$, normalized to total measure 1 (usually)
- Lebesgue measure on $\mathbb{R}$
Groups and Haar measure

Locally compact abelian groups:
- Integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- Finite cyclic group $\mathbb{Z}_m = \{0, 1, \ldots, m - 1\}$: addition mod $m$
- Reals $\mathbb{R}$
- Torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$: addition of reals mod 1

Haar measure on $G$ = translation invariant on $G$: $\mu(A) = \mu(A + t)$.
Unique up to scalar multiple.
- Counting measure on $\mathbb{Z}$
- Counting measure on $\mathbb{Z}_m$, normalized to total measure 1 (usually)
- Lebesgue measure on $\mathbb{R}$
- Lebesgue measure on $\mathbb{T}$ viewed as a circle
Groups and Haar measure

Locally compact abelian groups:

- Integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- Finite cyclic group $\mathbb{Z}_m = \{0, 1, \ldots, m - 1\}$: addition mod $m$
- Reals $\mathbb{R}$
- Torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$: addition of reals mod 1
- Products: $\mathbb{Z}^d$, $\mathbb{R}^d$, $\mathbb{T} \times \mathbb{R}$, etc

Haar measure on $G =$ translation invariant on $G$: $\mu(A) = \mu(A + t)$. Unique up to scalar multiple.

- Counting measure on $\mathbb{Z}$
- Counting measure on $\mathbb{Z}_m$, normalized to total measure 1 (usually)
- Lebesgue measure on $\mathbb{R}$
- Lebesgue measure on $\mathbb{T}$ viewed as a circle
- Product of Haar measures on the components
Character is a (continuous) group homomorphism from \( G \) to the \textit{multiplicative} group \( U = \{ z \in \mathbb{C} : |z| = 1 \} \).
Character is a (continuous) group homomorphism from $G$ to the \textit{multiplicative} group $U = \{ z \in \mathbb{C} : |z| = 1 \}$.

$\chi : G \to U$ satisfies $\chi(h + g) = \chi(h)\chi(g)$.

\begin{itemize}
  \item $G$ is compact $\iff \hat{\hat{G}}$ is discrete.
  \item Pontryagin duality: $\hat{\hat{G}} = G$.
\end{itemize}
Characters and the dual group

- **Character** is a (continuous) group homomorphism from $G$ to the multiplicative group $U = \{ z \in \mathbb{C} : |z| = 1 \}$.
- $\chi : G \rightarrow U$ satisfies $\chi(h + g) = \chi(h)\chi(g)$.
- If $\chi, \psi$ are characters then so is $\chi\psi$ (pointwise product). Write $\chi + \psi$ from now on instead of $\chi\psi$. 

Example: $G = T \times \mathbb{R} \Rightarrow \hat{G} = \mathbb{Z} \times \mathbb{R}$. The characters are $\chi_n, \chi_t(x, y) = \exp(2\pi i (nx + ty))$. 

$G$ is compact $\iff \hat{G}$ is discrete. 

Pontryagin duality: $\hat{\hat{G}} = G$. 

Mihalis Kolountzakis (U. of Crete)
Character is a (continuous) group homomorphism from \( G \) to the multiplicative group \( U = \{ z \in \mathbb{C} : |z| = 1 \} \).

\( \chi : G \to U \) satisfies \( \chi(h + g) = \chi(h)\chi(g) \)

If \( \chi, \psi \) are characters then so is \( \chi \psi \) (pointwise product). Write \( \chi + \psi \) from now on instead of \( \chi \psi \).

Group of characters (written additively) \( \hat{G} \) is the dual group of \( G \).
Characters and the dual group

- **Character** is a (continuous) group homomorphism from $G$ to the *multiplicative* group $U = \{z \in \mathbb{C} : |z| = 1\}$.
- $\chi : G \to U$ satisfies $\chi(h + g) = \chi(h)\chi(g)$
- If $\chi, \psi$ are characters then so is $\chi\psi$ (pointwise product). Write $\chi + \psi$ from now on instead of $\chi\psi$.
- Group of characters (written *additively*) $\hat{G}$ is the dual group of $G$
- $G = \mathbb{Z} \implies \hat{G} = \mathbb{T}$: the functions $\chi_x(n) = \exp(2\pi i nx), x \in \mathbb{T}$
Characters and the dual group

- **Character** is a (continuous) group homomorphism from $G$ to the *multiplicative* group $U = \{ z \in \mathbb{C} : |z| = 1 \}$.

- $\chi : G \to U$ satisfies $\chi(h + g) = \chi(h)\chi(g)$

- If $\chi, \psi$ are characters then so is $\chi\psi$ (pointwise product). Write $\chi + \psi$ from now on instead of $\chi\psi$.

- **Group of characters** (written *additively*) $\hat{G}$ is the dual group of $G$

- $G = \mathbb{Z} \implies \hat{G} = \mathbb{T}$: the functions $\chi_x(n) = \exp(2\pi inx), x \in \mathbb{T}$

- $G = \mathbb{T} \implies \hat{G} = \mathbb{Z}$: the functions $\chi_n(x) = \exp(2\pi inx), n \in \mathbb{Z}$

---

Mihalis Kolountzakis (U. of Crete)
Characters and the dual group

- **Character** is a (continuous) group homomorphism from $G$ to the *multiplicative* group $U = \{ z \in \mathbb{C} : |z| = 1 \}$.
- $\chi : G \rightarrow U$ satisfies $\chi(h + g) = \chi(h)\chi(g)$
- If $\chi, \psi$ are characters then so is $\chi \psi$ (pointwise product). Write $\chi + \psi$ from now on instead of $\chi \psi$.
- Group of characters (written *additively*) $\hat{G}$ is the *dual group* of $G$
  - $G = \mathbb{Z} \implies \hat{G} = \mathbb{T}$: the functions $\chi_x(n) = \exp(2\pi i xn), x \in \mathbb{T}$
  - $G = \mathbb{T} \implies \hat{G} = \mathbb{Z}$: the functions $\chi_n(x) = \exp(2\pi i nx), n \in \mathbb{Z}$
  - $G = \mathbb{R} \implies \hat{G} = \mathbb{R}$: the functions $\chi_t(x) = \exp(2\pi itx), t \in \mathbb{R}$

Pontryagin duality: $\hat{\hat{G}} = G$.
**Characters and the dual group**

- **Character** is a (continuous) group homomorphism from $G$ to the *multiplicative* group $U = \{ z \in \mathbb{C} : |z| = 1 \}$.

- $\chi : G \to U$ satisfies $\chi(h + g) = \chi(h)\chi(g)$

- If $\chi, \psi$ are characters then so is $\chi \psi$ (pointwise product). Write $\chi + \psi$ from now on instead of $\chi \psi$.

- Group of characters (written *additively*) $\hat{G}$ is the dual group of $G$

- $G = \mathbb{Z} \iff \hat{G} = \mathbb{T}$: the functions $\chi_x(n) = \exp(2\pi i nx), x \in \mathbb{T}$

- $G = \mathbb{T} \iff \hat{G} = \mathbb{Z}$: the functions $\chi_n(x) = \exp(2\pi inx), n \in \mathbb{Z}$

- $G = \mathbb{R} \iff \hat{G} = \mathbb{R}$: the functions $\chi_t(x) = \exp(2\pi i tx), t \in \mathbb{R}$

- $G = \mathbb{Z}_m \iff \hat{G} = \mathbb{Z}_m$: the functions $\chi_k(n) = \exp(2\pi i kn/m), k \in \mathbb{Z}_m$
Character is a (continuous) group homomorphism from $G$ to the multiplicative group $U = \{z \in \mathbb{C} : |z| = 1\}$. 

- $\chi : G \rightarrow U$ satisfies $\chi(h + g) = \chi(h)\chi(g)$ 
- If $\chi, \psi$ are characters then so is $\chi \psi$ (pointwise product). Write $\chi + \psi$ from now on instead of $\chi \psi$. 

Group of characters (written additively) $\hat{G}$ is the dual group of $G$ 

- $G = \mathbb{Z} \implies \hat{G} = \mathbb{T}$: the functions $\chi_x(n) = \exp(2\pi ixn), x \in \mathbb{T}$ 
- $G = \mathbb{T} \implies \hat{G} = \mathbb{Z}$: the functions $\chi_n(x) = \exp(2\pi inx), n \in \mathbb{Z}$ 
- $G = \mathbb{R} \implies \hat{G} = \mathbb{R}$: the functions $\chi_t(x) = \exp(2\pi itx), t \in \mathbb{R}$ 
- $G = \mathbb{Z}_m \implies \hat{G} = \mathbb{Z}_m$: the functions $\chi_k(n) = \exp(2\pi ikn/m), k \in \mathbb{Z}_m$ 
- $G = A \times B \implies \hat{G} = \hat{A} \times \hat{B}$
Character is a (continuous) group homomorphism from $G$ to the _multiplicative_ group $U = \{z \in \mathbb{C} : |z| = 1\}$.

$\chi : G \to U$ satsifies $\chi(h + g) = \chi(h)\chi(g)$

If $\chi, \psi$ are characters then so is $\chi\psi$ (pointwise product). Write $\chi + \psi$ from now on instead of $\chi\psi$.

Group of characters (written _additively_) $\hat{G}$ is the dual group of $G$

- $G = \mathbb{Z} \implies \hat{G} = \mathbb{T}$: the functions $\chi_x(n) = \exp(2\pi inx), x \in \mathbb{T}$
- $G = \mathbb{T} \implies \hat{G} = \mathbb{Z}$: the functions $\chi_n(x) = \exp(2\pi inx), n \in \mathbb{Z}$
- $G = \mathbb{R} \implies \hat{G} = \mathbb{R}$: the functions $\chi_t(x) = \exp(2\pi itx), t \in \mathbb{R}$
- $G = \mathbb{Z}_m \implies \hat{G} = \mathbb{Z}_m$: the functions $\chi_k(n) = \exp(2\pi ikn/m), k \in \mathbb{Z}_m$
- $G = A \times B \implies \hat{G} = \hat{A} \times \hat{B}$
- Example: $G = \mathbb{T} \times \mathbb{R} \implies \hat{G} = \mathbb{Z} \times \mathbb{R}$. The characters are $\chi_{n,t}(x, y) = \exp(2\pi i(nx + ty))$. 
Character is a (continuous) group homomorphism from $G$ to the multiplicative group $U = \{z \in \mathbb{C} : |z| = 1\}$.

- $\chi : G \to U$ satisfies $\chi(h + g) = \chi(h)\chi(g)$
- If $\chi, \psi$ are characters then so is $\chi \psi$ (pointwise product). Write $\chi + \psi$ from now on instead of $\chi \psi$.

Group of characters (written additively) $\hat{G}$ is the dual group of $G$

- $G = \mathbb{Z} \Rightarrow \hat{G} = \mathbb{T}$: the functions $\chi_x(n) = \exp(2\pi i nx), x \in \mathbb{T}$
- $G = \mathbb{T} \Rightarrow \hat{G} = \mathbb{Z}$: the functions $\chi_n(x) = \exp(2\pi inx), n \in \mathbb{Z}$
- $G = \mathbb{R} \Rightarrow \hat{G} = \mathbb{R}$: the functions $\chi_t(x) = \exp(2\pi itx), t \in \mathbb{R}$
- $G = \mathbb{Z}_m \Rightarrow \hat{G} = \mathbb{Z}_m$: the functions $\chi_k(n) = \exp(2\pi i kn/m), k \in \mathbb{Z}_m$
- $G = A \times B \Rightarrow \hat{G} = \hat{A} \times \hat{B}$

Example: $G = \mathbb{T} \times \mathbb{R} \Rightarrow \hat{G} = \mathbb{Z} \times \mathbb{R}$. The characters are $\chi_{n,t}(x, y) = \exp(2\pi i(nx + ty))$.

- $G$ is compact $\iff \hat{G}$ is discrete
Characters and the dual group

- **Character** is a (continuous) group homomorphism from $G$ to the *multiplicative* group $U = \{ z \in \mathbb{C} : |z| = 1 \}$.
- $\chi : G \to U$ satisfies $\chi(h + g) = \chi(h)\chi(g)$
- If $\chi, \psi$ are characters then so is $\chi \psi$ (pointwise product). Write $\chi + \psi$ from now on instead of $\chi \psi$.
- Group of characters (written *additively*) $\hat{G}$ is the dual group of $G$
  - $G = \mathbb{Z} \iff \hat{G} = \mathbb{T}$: the functions $\chi_x(n) = \exp(2\pi i x n)$, $x \in \mathbb{T}$
  - $G = \mathbb{T} \iff \hat{G} = \mathbb{Z}$: the functions $\chi_n(x) = \exp(2\pi i n x)$, $n \in \mathbb{Z}$
  - $G = \mathbb{R} \iff \hat{G} = \mathbb{R}$: the functions $\chi_t(x) = \exp(2\pi i t x)$, $t \in \mathbb{R}$
  - $G = \mathbb{Z}_m \iff \hat{G} = \mathbb{Z}_m$: the functions $\chi_k(n) = \exp(2\pi i k n / m)$, $k \in \mathbb{Z}_m$
  - $G = A \times B \iff \hat{G} = \hat{A} \times \hat{B}$
- Example: $G = \mathbb{T} \times \mathbb{R} \iff \hat{G} = \mathbb{Z} \times \mathbb{R}$. The characters are $\chi_{n,t}(x, y) = \exp(2\pi i (nx + ty))$.
- $G$ is compact $\iff \hat{G}$ is discrete
- **Pontryagin** duality: $\hat{\hat{G}} = G$. 

Mihalis Kolountzakis (U. of Crete)
The Fourier Transform of integrable functions

- \( f \in L^1(G) \). That is \( \| f \|_1 := \int_G |f(x)| \, d\mu(x) < \infty \)
The Fourier Transform of integrable functions

- $f \in L^1(G)$. That is $\|f\|_1 := \int_G |f(x)| \, d\mu(x) < \infty$
- If $G$ is finite then $L^1(G)$ is all functions $G \to \mathbb{C}$
The Fourier Transform of integrable functions

- $f \in L^1(G)$. That is $\|f\|_1 := \int_G |f(x)| \, d\mu(x) < \infty$
- If $G$ is finite then $L^1(G)$ is all functions $G \to \mathbb{C}$
- The FT of $f$ is $\hat{f} : \hat{G} \to \mathbb{C}$ defined by

$$\hat{f}(\chi) = \int_G f(x) \overline{\chi(x)} \, d\mu(x), \quad \chi \in \hat{G}$$
The Fourier Transform of integrable functions

- \( f \in L^1(G) \). That is \( \|f\|_1 := \int_G |f(x)| \, d\mu(x) < \infty \)
- If \( G \) is finite then \( L^1(G) \) is all functions \( G \to \mathbb{C} \)
- The FT of \( f \) is \( \hat{f} : \hat{G} \to \mathbb{C} \) defined by
  \[
  \hat{f}(\chi) = \int_G f(x) \overline{\chi(x)} \, d\mu(x), \quad \chi \in \hat{G}
  \]
- Example: \( G = \mathbb{T} \) ("Fourier coefficients"): \[
  \hat{f}(n) = \int_{\mathbb{T}} f(x) e^{-2\pi i nx} \, dx, \quad n \in \mathbb{Z}
  \]
The Fourier Transform of integrable functions

- $f \in L^1(G)$. That is $\|f\|_1 := \int_G |f(x)| \, d\mu(x) < \infty$
- If $G$ is finite then $L^1(G)$ is all functions $G \to \mathbb{C}$
- The FT of $f$ is $\hat{f} : \hat{G} \to \mathbb{C}$ defined by

$$\hat{f}(\chi) = \int_G f(x) \overline{\chi(x)} \, d\mu(x), \quad \chi \in \hat{G}$$

- Example: $G = \mathbb{T}$ ("Fourier coefficients"): 
  $$\hat{f}(n) = \int_{\mathbb{T}} f(x) e^{-2\pi i nx} \, dx, \quad n \in \mathbb{Z}$$
- Example: $G = \mathbb{R}$ ("Fourier transform"): 
  $$\hat{f}(\xi) = \int_{\mathbb{T}} f(x) e^{-2\pi i \xi x} \, dx, \quad \xi \in \mathbb{R}$$
The Fourier Transform of integrable functions

- $f \in L^1(G)$. That is \( \|f\|_1 := \int_G |f(x)| \, d\mu(x) < \infty \)
- If $G$ is finite then $L^1(G)$ is all functions $G \to \mathbb{C}$
- The FT of $f$ is $\hat{f} : \hat{G} \to \mathbb{C}$ defined by

  \[ \hat{f}(\chi) = \int_G f(x)\overline{\chi(x)} \, d\mu(x), \quad \chi \in \hat{G} \]

- Example: $G = \mathbb{T}$ ("Fourier coefficients"): \[ \hat{f}(n) = \int_{\mathbb{T}} f(x)e^{-2\pi inx} \, dx, \quad n \in \mathbb{Z} \]
- Example: $G = \mathbb{R}$ ("Fourier transform"): \[ \hat{f}(\xi) = \int_{\mathbb{T}} f(x)e^{-2\pi i\xi x} \, dx, \quad \xi \in \mathbb{R} \]
- Example: $G = \mathbb{Z}_m$ ("Discrete Fourier transform or DFT"): \[ \hat{f}(k) = \frac{1}{m} \sum_{j=0}^{m-1} f(j)e^{-2\pi ikj/m}, \quad k \in \mathbb{Z}_m \]
Elementary properties of the Fourier Transform

- **Linearity**: 
  \[ \hat{\lambda f + \mu g} = \lambda \hat{f} + \mu \hat{g}. \]

- **Symmetry**: 
  \[ \hat{f}(\xi) = \hat{f}(-\xi), \quad \hat{\xi f}(x) = \hat{f}(-\xi) \]

  If \( f \) is real, then 
  \[ \hat{f}(\xi) = \hat{f}(-\xi) \]

- **Translation**: 
  If \( \tau \in G \), \( \xi \in \hat{G} \), 
  \[ f_{\tau}(x) = f(x - \tau) \] then 
  \[ \hat{f}_\tau(\xi) = \xi(\tau) \cdot \hat{f}(\xi) \]

- **Example**: 
  \( G = T \): 
  \[ \hat{f}(x-\theta)(n) = e^{-2\pi i \theta n} \hat{f}(n), \quad \theta \in T, \quad n \in \mathbb{Z} \]

- **Modulation**: 
  If \( \chi, \xi \in \hat{G} \) then 
  \[ \hat{\chi f}(x) = \hat{f}(\xi - \chi) \]

- **Example**: 
  \( G = \mathbb{R} \): 
  \[ \hat{e^{2\pi i tx}} f(x)(\xi) = \hat{f}(\xi - t) \]
Elementary properties of the Fourier Transform

- **Linearity:** \( \lambda f + \mu g = \lambda \hat{f} + \mu \hat{g} \).
- **Symmetry:** \( \hat{f}(-x) = \hat{f}(x) \), \( \hat{f}(x) = \hat{f}(-x) \)
Elementary properties of the Fourier Transform

- **Linearity:** \( \lambda f + \mu g = \lambda \hat{f} + \mu \hat{g} \).
- **Symmetry:** \( \hat{f}(-x) = \overline{\hat{f}(x)} \), \( \hat{f}(x) = \overline{\hat{f}(-x)} \)
- **Real \( f \):** then \( \hat{f}(x) = \overline{\hat{f}(-x)} \)
Elementary properties of the Fourier Transform

- **Linearity:** \( \lambda \hat{f} + \mu \hat{g} = \hat{\lambda f} + \hat{\mu g} \).
- **Symmetry:** \( \hat{f}(-x) = \hat{f}(x), \hat{f}(x) = \hat{f}(-x) \)
- **Real \( f \):** then \( \hat{f}(x) = \hat{f}(-x) \)
- **Translation:** if \( \tau \in G, \xi \in \hat{G} \), \( f_\tau(x) = f(x - \tau) \) then \( \hat{f}_\tau(\xi) = \xi(\tau) \cdot \hat{f}(\xi) \).

Example: \( G = \mathbb{T} \): \( f(x - \theta)(n) = e^{-2\pi in\theta} \hat{f}(n) \), for \( \theta \in \mathbb{T}, n \in \mathbb{Z} \).
Elementary properties of the Fourier Transform

- **Linearity:** \( \lambda \hat{f} + \mu \hat{g} = \hat{\lambda f + \mu g} \).
- **Symmetry:** \( \hat{f}(-x) = \hat{\bar{f}}(x) \), \( \bar{f}(x) = \hat{f}(-x) \).
- **Real \( f \):** then \( \hat{f}(x) = \hat{\bar{f}}(-x) \).
- **Translation:** if \( \tau \in G, \xi \in \hat{G} \), \( f_\tau(x) = f(x - \tau) \) then \( \hat{f}_\tau(\xi) = \bar{\xi}(\tau) \cdot \hat{f}(\xi) \).
  
  **Example:** \( G = \mathbb{T} \): \( f(x - \theta)(n) = e^{-2\pi i n \theta} \hat{f}(n) \), for \( \theta \in \mathbb{T}, n \in \mathbb{Z} \).
- **Modulation:** If \( \chi, \xi \in \hat{G} \) then \( \chi(x)f(x)(\xi) = \hat{f}(\xi - \chi) \).
  
  **Example:** \( G = \mathbb{R} \): \( e^{2\pi i t x} f(x)(\xi) = \hat{f}(\xi - t) \).
Elementary properties of the Fourier Transform

- **Linearity:** \( \lambda \hat{f} + \mu \hat{g} = \lambda \hat{f} + \mu \hat{g} \).
- **Symmetry:** \( \hat{f}(-x) = \hat{f}(x) \), \( \hat{f}(x) = \hat{f}(-x) \)
- **Real** \( f \): then \( \hat{f}(x) = \hat{f}(-x) \)
- **Translation:** if \( \tau \in G, \xi \in \hat{G} \), \( f_\tau(x) = f(x - \tau) \) then \( \hat{f}_\tau(\xi) = \xi(\tau) \cdot \hat{f}(\xi) \).
  
  *Example:* \( G = \mathbb{T} \): \( f(x - \theta)(n) = e^{-2\pi in\theta} \hat{f}(n) \), for \( \theta \in \mathbb{T}, n \in \mathbb{Z} \).
- **Modulation:** If \( \chi, \xi \in \hat{G} \) then \( \chi(x)f(x)(\xi) = \hat{f}(\xi - \chi) \).
  
  *Example:* \( G = \mathbb{R} \): \( e^{2\pi itx}f(x)(\xi) = \hat{f}(\xi - t) \).
- \( f, g \in L^1(G) \): their **convolution** is \( f * g(x) = \int_G f(t)g(x - t) \, d\mu(t) \).
  
  Then \( \|f * g\|_1 \leq \|f\|_1 \|g\|_1 \) and
  
  \[ \hat{f} \hat{g}(\xi) = \hat{f}(\xi) \cdot \hat{g}(\xi), \quad \xi \in \hat{G} \]
If $G$ is compact ($\implies$ total Haar measure $= 1$) then characters are in $L^1(G)$, being bounded.
Orthogonality of characters on compact groups

- If $G$ is compact ($\iff$ total Haar measure $= 1$) then characters are in $L^1(G)$, being bounded.
- If $\chi \in \widehat{G}$ then
  \[
  \int_G \chi(x) \, dx = \int_G \chi(x + g) \, dx = \chi(g) \int_G \chi(x) \, dx,
  \]
  so $\int_G \chi = 0$ if $\chi$ nontrivial, $1$ if $\chi$ is trivial ($= 1$).
If $G$ is compact (⇒ total Haar measure = 1) then characters are in $L^1(G)$, being bounded.

If $\chi \in \hat{G}$ then

$$\int_G \chi(x) \, dx = \int_G \chi(x + g) \, dx = \chi(g) \int_G \chi(x) \, dx,$$

so $\int_G \chi = 0$ if $\chi$ nontrivial, 1 if $\chi$ is trivial (= 1).

If $\chi, \psi \in \hat{G}$ then $\chi(x)\psi(-x)$ is also a character. Hence

$$\langle \chi, \psi \rangle = \int_G \chi(x)\overline{\psi(x)} \, dx = \int_G \chi(x)\psi(-x) \, dx = \begin{cases} 1 & \chi = \psi \\ 0 & \chi \neq \psi \end{cases}$$
Orthogonality of characters on compact groups

- If $G$ is compact ($\Rightarrow$ total Haar measure $= 1$) then characters are in $L^1(G)$, being bounded.

- If $\chi \in \hat{G}$ then

  $$\int_G \chi(x) \, dx = \int_G \chi(x + g) \, dx = \chi(g) \int_G \chi(x) \, dx,$$

so $\int_G \chi = 0$ if $\chi$ nontrivial, 1 if $\chi$ is trivial ($= 1$).

- If $\chi, \psi \in \hat{G}$ then $\chi(x)\psi(-x)$ is also a character. Hence

  $$\langle \chi, \psi \rangle = \int_G \chi(x)\overline{\psi(x)} \, dx = \int_G \chi(x)\psi(-x) \, dx = \begin{cases} 1 & \chi = \psi \\ 0 & \chi \neq \psi \end{cases}$$

- Fourier representation (inversion) in $\mathbb{Z}_m$: $G = \mathbb{Z}_m \implies$ the $m$ characters form a complete orthonormal set in $L^2(G)$:

  $$f(x) = \sum_{k=0}^{m-1} \langle f(\cdot), e^{2\pi ik\cdot} \rangle e^{2\pi ikx} = \sum_{k=0}^{m-1} \hat{f}(k)e^{2\pi ikx}$$
$L^2$ of compact $G$

- Trigonometric polynomials = finite linear combinations of characters on $G$
$L^2$ of compact $G$

- **Trigonometric polynomials** = finite linear combinations of characters on $G$

- Example: $G = \mathbb{T}$. Trig. polynomials are of the type $\sum_{k=-N}^{N} c_k e^{2\pi ikx}$. The least such $N$ is called the **degree** of the polynomial.
$L^2$ of compact $G$

- **Trigonometric polynomials** = finite linear combinations of characters on $G$
- **Example:** $G = \mathbb{T}$. Trig. polynomials are of the type $\sum_{k=-N}^{N} c_k e^{2\pi i kx}$. The least such $N$ is called the **degree** of the polynomial.
- **Example:** $G = \mathbb{R}$. Trig. polynomials are of the type $\sum_{k=1}^{K} c_k e^{2\pi i \lambda_k x}$, where $\lambda_j \in \mathbb{R}$. 
$L^2$ of compact $G$

- **Trigonometric polynomials** = finite linear combinations of characters on $G$

  - Example: $G = \mathbb{T}$. Trig. polynomials are of the type $\sum_{k=-N}^{N} c_k e^{2\pi i k x}$. The least such $N$ is called the degree of the polynomial.

  - Example: $G = \mathbb{R}$. Trig. polynomials are of the type $\sum_{k=1}^{K} c_k e^{2\pi i \lambda_k x}$, where $\lambda_j \in \mathbb{R}$.

  - **Compact $G$: Stone - Weierstrass Theorem $\implies$ trig. polynomials dense in $C(G)$ (in $\| \cdot \|_\infty$).
**$L^2$ of compact $G$**

- **Trigonometric polynomials** = finite linear combinations of characters on $G$

- Example: $G = \mathbb{T}$. Trig. polynomials are of the type $\sum_{k=-N}^{N} c_k e^{2\pi i k x}$. The least such $N$ is called the degree of the polynomial.

- Example: $G = \mathbb{R}$. Trig. polynomials are of the type $\sum_{k=1}^{K} c_k e^{2\pi i \lambda_k x}$, where $\lambda_j \in \mathbb{R}$.

- **Compact $G$: Stone - Weierstrass Theorem $\implies$ trig. polynomials dense in $C(G)$ (in $\| \cdot \|_\infty$).

- **Fourier representation in $L^2(G)$**: Compact $G$: The characters form a complete ONS. Since $C(G)$ is dense in $L^2(G)$:

  $$ f = \int_{\chi \in \hat{G}} \hat{f}(\chi) \chi \, d\chi \quad \text{all} \ f \in L^2(G), \ \text{convergence in} \ L^2(G) $$
Trigonometric polynomials = finite linear combinations of characters on $G$

Example: $G = \mathbb{T}$. Trig. polynomials are of the type $\sum_{k=-N}^{N} c_k e^{2\pi i k x}$. The least such $N$ is called the degree of the polynomial.

Example: $G = \mathbb{R}$. Trig. polynomials are of the type $\sum_{k=1}^{K} c_k e^{2\pi i \lambda_k x}$, where $\lambda_j \in \mathbb{R}$.

Compact $G$: Stone - Weierstrass Theorem $\implies$ trig. polynomials dense in $C(G)$ (in $\| \cdot \|_\infty$).

Fourier representation in $L^2(G)$: Compact $G$: The characters form a complete ONS. Since $C(G)$ is dense in $L^2(G)$:

$$f = \int_{\hat{G}} \hat{f}(\chi) \chi \, d\chi \quad \text{all } f \in L^2(G), \text{ convergence in } L^2(G)$$

$\hat{G}$ necessarily discrete in this case
**L^2 of compact G, continued**

- **Compact G**: Parseval formula:

\[ \int_G f(x) \overline{g(x)} \, dx = \int_{\hat{G}} \hat{f}(\chi) \overline{\hat{g}(\chi)} \, d\chi. \]
Compact $G$: Parseval formula:

$$\int_{G} f(x)\overline{g(x)} \, dx = \int_{\hat{G}} \hat{f}(\chi)\overline{\hat{g}(\chi)} \, d\chi.$$  

Compact $G$: $f \rightarrow \hat{f}$ is an isometry from $L^2(G)$ onto $L^2(\hat{G})$. 
$L^2$ of compact $G$, continued

- Compact $G$: Parseval formula:
  \[
  \int_G f(x)g(x) \, dx = \int_{\hat{G}} \hat{f}(\chi)\hat{g}(\chi) \, d\chi.
  \]

- Compact $G$: $f \mapsto \hat{f}$ is an isometry from $L^2(G)$ onto $L^2(\hat{G})$.

- Example: $G = \mathbb{T}$
  \[
  \int_{\mathbb{T}} f(x)g(x) \, dx = \sum_{k \in \mathbb{Z}} \hat{f}(k)\hat{g}(k), \quad f, g \in L^2(\mathbb{T}).
  \]
$L^2$ of compact $G$, continued

- Compact $G$: Parseval formula:

  $$\int_G f(x)\overline{g(x)} \, dx = \int_{\hat{G}} \widehat{f}(\chi)\overline{\widehat{g}(\chi)} \, d\chi.$$  

- Compact $G$: $f \mapsto \widehat{f}$ is an isometry from $L^2(G)$ onto $L^2(\hat{G})$.

- Example: $G = \mathbb{T}$

  $$\int_{\mathbb{T}} f(x)\overline{g(x)} \, dx = \sum_{k \in \mathbb{Z}} \widehat{f}(k)\overline{\widehat{g}(k)}, \quad f, g \in L^2(\mathbb{T}).$$

- Example: $G = \mathbb{Z}_m$

  $$\sum_{j=0}^{m-1} f(j)\overline{g(j)} = \sum_{k=0}^{m-1} \widehat{f}(k)\overline{\widehat{g}(k)}, \quad \text{all } f, g : \mathbb{Z}_m \to \mathbb{C}.$$
Problem of significance in (a) crystallography, (b) astrophysics: determine a subset $E \subseteq \mathbb{Z}_n$ from its triple correlation:

$$N_E(a, b) = \#\{x \in \mathbb{Z}_n : x, x + a, x + b \in E\}, \quad a, b \in \mathbb{Z}_n$$

$$= \sum_{x \in \mathbb{Z}_n} 1_E(x)1_E(x + a)1_E(x + b)$$

Counts number of occurrences of translated 3-point patterns $\{0, a, b\}$. 
Triple correlations in $\mathbb{Z}_p$: an application

- Problem of significance in (a) crystallography, (b) astrophysics: determine a subset $E \subseteq \mathbb{Z}_n$ from its triple correlation:

$$N_E(a, b) = \# \{ x \in \mathbb{Z}_n : x, x + a, x + b \in E \}, \quad a, b \in \mathbb{Z}_n$$

$$= \sum_{x \in \mathbb{Z}_n} 1_E(x)1_E(x + a)1_E(x + b)$$

Counts number of occurrences of translated 3-point patterns $\{0, a, b\}$.

- $E$ can only be determined up to translation: $E$ and $E + t$ have the same $N(\cdot, \cdot)$. 

Fourier transform of $N_E$:

$$\hat{N}_E(\xi, \eta) = \hat{1}_E(\xi)\hat{1}_E(\eta)\hat{1}_E(- (\xi + \eta)), \quad \xi, \eta \in \mathbb{Z}_n.$$
Problem of significance in (a) crystallography, (b) astrophysics: determine a subset $E \subseteq \mathbb{Z}_n$ from its triple correlation:

$$N_E(a, b) = \# \{ x \in \mathbb{Z}_n : x, x+a, x+b \in E \}, \quad a, b \in \mathbb{Z}_n$$

$$= \sum_{x \in \mathbb{Z}_n} 1_E(x) 1_E(x+a) 1_E(x+b)$$

Counts number of occurrences of translated 3-point patterns $\{0, a, b\}$.

$E$ can only be determined up to translation: $E$ and $E + t$ have the same $N(\cdot, \cdot)$.

For general $n$ it has been proved that $N(\cdot, \cdot)$ cannot determine $E$ even up to translation (non-trivial).
Problem of significance in (a) crystallography, (b) astrophysics: determine a subset $E \subset \mathbb{Z}_n$ from its triple correlation:

$$N_E(a, b) = \# \{ x \in \mathbb{Z}_n : x, x + a, x + b \in E \}, \quad a, b \in \mathbb{Z}_n$$

$$= \sum_{x \in \mathbb{Z}_n} 1_E(x) 1_E(x + a) 1_E(x + b)$$

Counts number of occurrences of translated 3-point patterns $\{0, a, b\}$.

$E$ can only be determined up to translation: $E$ and $E + t$ have the same $N(\cdot, \cdot)$.

For general $n$ it has been proved that $N(\cdot, \cdot)$ cannot determine $E$ even up to translation (non-trivial).

Special case: $E$ can be determined up to translation from $N(\cdot, \cdot)$ if $n = p$ is a prime.
Problem of significance in (a) crystallography, (b) astrophysics: determine a subset $E \subseteq \mathbb{Z}_n$ from its triple correlation:

$$N_E(a, b) = \# \{x \in \mathbb{Z}_n : x, x + a, x + b \in E\}, \ a, b \in \mathbb{Z}_n$$

$$= \sum_{x \in \mathbb{Z}_n} 1_E(x)1_E(x + a)1_E(x + b)$$

Counts number of occurrences of translated 3-point patterns $\{0, a, b\}$.

$E$ can only be determined up to translation: $E$ and $E + t$ have the same $N(\cdot, \cdot)$.

For general $n$ it has been proved that $N(\cdot, \cdot)$ cannot determine $E$ even up to translation (non-trivial).

Special case: $E$ can be determined up to translation from $N(\cdot, \cdot)$ if $n = p$ is a prime.

Fourier transform of $N_E : \mathbb{Z}_n \times \mathbb{Z}_n \rightarrow \mathbb{R}$ is easily computed:

$$\hat{N}_E(\xi, \eta) = \hat{1}_E(\xi)\hat{1}_E(\eta)\hat{1}_E(-(\xi + \eta)), \ \xi, \eta \in \mathbb{Z}_n.$$
If \( N_E \equiv N_F \) for \( E, F \subseteq \mathbb{Z}_n \) then

\[
\hat{1}_E(\xi)\hat{1}_E(\eta)\hat{1}_E(-\xi - \eta) = \hat{1}_F(\xi)\hat{1}_F(\eta)\hat{1}_F(-\xi - \eta), \quad \xi, \eta \in \mathbb{Z}_n \tag{1}
\]
If $N_E \equiv N_F$ for $E, F \subseteq \mathbb{Z}_n$ then

$$\hat{1}_E(\xi)\hat{1}_E(\eta)\hat{1}_E(-(\xi + \eta)) = \hat{1}_F(\xi)\hat{1}_F(\eta)\hat{1}_F(-(\xi + \eta)), \quad \xi, \eta \in \mathbb{Z}_n \ (1)$$

Setting $\xi = \eta = 0$ we deduce $\#E = \#F$. 

Hence $\phi : \mathbb{Z}_n \to \mathbb{C}$ is a character and $\hat{1}_E \equiv \phi \hat{1}_F$. 

Since $\hat{1}_{\mathbb{Z}_n} = \mathbb{Z}_n$ we have $\phi(\xi) = e^{2\pi i t\xi/n}$ for some $t \in \mathbb{Z}_n$. 

Hence $E = F + t$ so $N_E$ determines $E$ up to translation if $\hat{1}_E$ is never 0.
If $N_E \equiv N_F$ for $E, F \subseteq \mathbb{Z}_n$ then
\[
\hat{1}_E(\xi)\hat{1}_E(\eta)\hat{1}_E(-(\xi + \eta)) = \hat{1}_F(\xi)\hat{1}_F(\eta)\hat{1}_F(-(\xi + \eta)), \quad \xi, \eta \in \mathbb{Z}_n \quad (1)
\]

- Setting $\xi = \eta = 0$ we deduce $\#E = \#F$.
- Setting $\eta = 0$, and using $\hat{f}(-x) = \overline{\hat{f}(x)}$ for real $f$, we get $|\hat{1}_E| \equiv |\hat{1}_F|$. 

Hence $\phi: \mathbb{Z}_n \rightarrow \mathbb{C}$ is a character and $\hat{1}_E \equiv \phi\hat{1}_F$.
Triple correlations in $\mathbb{Z}_p$: an application (continued)

- If $N_E \equiv N_F$ for $E, F \subseteq \mathbb{Z}_n$ then

\[
\hat{1}_E(\xi)\hat{1}_E(\eta)\hat{1}_E(-(\xi+\eta)) = \hat{1}_F(\xi)\hat{1}_F(\eta)\hat{1}_F(-(\xi+\eta)), \quad \xi, \eta \in \mathbb{Z}_n \quad (1)
\]

- Setting $\xi = \eta = 0$ we deduce $\#E = \#F$.
- Setting $\eta = 0$, and using $\hat{f}(-x) = \overline{\hat{f}(x)}$ for real $f$, we get $|\hat{1}_E| \equiv |\hat{1}_F|$.
- If $\hat{1}_F$ is never 0 we divide (1) by its RHS to get

\[
\phi(\xi)\phi(\eta) = \phi(\xi + \eta), \quad \text{where } \phi = \frac{\hat{1}_E}{\hat{1}_F} \quad (2)
\]
If \( N_E \equiv N_F \) for \( E, F \subseteq \mathbb{Z}_n \) then
\[
\hat{1}_E(\xi)\hat{1}_E(\eta)\hat{1}_E(-(\xi+\eta)) = \hat{1}_F(\xi)\hat{1}_F(\eta)\hat{1}_F(-(\xi+\eta)), \ \xi, \eta \in \mathbb{Z}_n \quad (1)
\]

- Setting \( \xi = \eta = 0 \) we deduce \( \#E = \#F \).
- Setting \( \eta = 0 \), and using \( \hat{f}(-x) = \overline{\hat{f}(x)} \) for real \( f \), we get \( \left| \hat{1}_E \right| = \left| \hat{1}_F \right| \).
- If \( \hat{1}_F \) is never 0 we divide (1) by its RHS to get
\[
\phi(\xi)\phi(\eta) = \phi(\xi + \eta), \ \text{where} \ \phi = \frac{\hat{1}_E}{\hat{1}_F} \quad (2)
\]

- Hence \( \phi : \mathbb{Z}_n \rightarrow \mathbb{C} \) is a character and \( \hat{1}_E \equiv \phi\hat{1}_F \).
If \( N_E \equiv N_F \) for \( E, F \subseteq \mathbb{Z}_n \) then

\[
\hat{1}_E(\xi)\hat{1}_E(\eta)\hat{1}_E(-(\xi + \eta)) = \hat{1}_F(\xi)\hat{1}_F(\eta)\hat{1}_F(-(\xi + \eta)), \quad \xi, \eta \in \mathbb{Z}_n \tag{1}
\]

Setting \( \xi = \eta = 0 \) we deduce \( \#E = \#F \).

Setting \( \eta = 0 \), and using \( \hat{f}(-x) = \overline{\hat{f}(x)} \) for real \( f \), we get \( \left| \hat{1}_E \right| \equiv \left| \hat{1}_F \right| \).

If \( \hat{1}_F \) is never 0 we divide (1) by its RHS to get

\[
\phi(\xi)\phi(\eta) = \phi(\xi + \eta), \quad \text{where } \phi = \frac{\hat{1}_E}{\hat{1}_F} \tag{2}
\]

Hence \( \phi : \mathbb{Z}_n \rightarrow \mathbb{C} \) is a character and \( \hat{1}_E \equiv \phi \hat{1}_F \).

Since \( \hat{\mathbb{Z}}_n = \mathbb{Z}_n \) we have \( \phi(\xi) = e^{2\pi it\xi/n} \) for some \( t \in \mathbb{Z}_n \).
If \( N_E \equiv N_F \) for \( E, F \subseteq \mathbb{Z}_n \) then
\[
\hat{1}_E(\xi)\hat{1}_E(\eta)\hat{1}_E(-(\xi + \eta)) = \hat{1}_F(\xi)\hat{1}_F(\eta)\hat{1}_F(-(\xi + \eta)), \quad \xi, \eta \in \mathbb{Z}_n \quad (1)
\]

- Setting \( \xi = \eta = 0 \) we deduce \( \#E = \#F \).
- Setting \( \eta = 0 \), and using \( \hat{f}(-x) = \overline{\hat{f}(x)} \) for real \( f \), we get \( |\hat{1}_E| = |\hat{1}_F| \).
- If \( \hat{1}_F \) is never 0 we divide (1) by its RHS to get
\[
\phi(\xi)\phi(\eta) = \phi(\xi + \eta), \quad \text{where} \quad \phi = \frac{\hat{1}_E}{\hat{1}_F} \quad (2)
\]

- Hence \( \phi : \mathbb{Z}_n \to \mathbb{C} \) is a character and \( \hat{1}_E \equiv \phi \hat{1}_F \).
- Since \( \mathbb{Z}_n = \mathbb{Z}_n \) we have \( \phi(\xi) = e^{2\pi it\xi/n} \) for some \( t \in \mathbb{Z}_n \).
- Hence \( E = F + t \).
If $N_E \equiv N_F$ for $E, F \subseteq \mathbb{Z}_n$ then

$$\hat{1}_E(\xi)\hat{1}_E(\eta)\hat{1}_E(-(\xi + \eta)) = \hat{1}_F(\xi)\hat{1}_F(\eta)\hat{1}_F(-(\xi + \eta)), \quad \xi, \eta \in \mathbb{Z}_n \quad (1)$$

- Setting $\xi = \eta = 0$ we deduce $\#E = \#F$.
- Setting $\eta = 0$, and using $\hat{f}(-x) = \overline{\hat{f}(x)}$ for real $f$, we get $|\hat{1}_E| \equiv |\hat{1}_F|$.
- If $\hat{1}_F$ is never 0 we divide (1) by its RHS to get

$$\phi(\xi)\phi(\eta) = \phi(\xi + \eta), \quad \text{where } \phi = \frac{\hat{1}_E}{\hat{1}_F} \quad (2)$$

- Hence $\phi : \mathbb{Z}_n \to \mathbb{C}$ is a character and $\hat{1}_E \equiv \phi \hat{1}_F$.
- Since $\hat{\mathbb{Z}}_n = \mathbb{Z}_n$ we have $\phi(\xi) = e^{2\pi it\xi/n}$ for some $t \in \mathbb{Z}_n$.
- Hence $E = F + t$.
- So $N_E$ determines $E$ up to translation if $\hat{1}_E$ is never 0.
Suppose \( n = p \) is a prime, \( E \subseteq \mathbb{Z}_p \). Then

\[
\widehat{1_E}(\xi) = \frac{1}{p} \sum_{s \in E} (\zeta \xi)^s, \quad \zeta = e^{-2\pi i/p} \text{ is a } p\text{-root of unity.} \tag{3}
\]
Suppose $n = p$ is a prime, $E \subseteq \mathbb{Z}_p$. Then

$$\mathbf{1}_E(\xi) = \frac{1}{p} \sum_{s \in E} (\zeta^\xi)^s, \quad \zeta = e^{-2\pi i/p} \text{ is a } p\text{-root of unity.} \quad (3)$$

Each $\zeta^\xi, \xi \neq 0$, is a primitive $p$-th root of unity itself.
Triple correlations in $\mathbb{Z}_p$: an application (conclusion)

- Suppose $n = p$ is a prime, $E \subseteq \mathbb{Z}_p$. Then

$$\hat{1}_E(\xi) = \frac{1}{p} \sum_{s \in E} (\zeta^s)^s, \quad \zeta = e^{-2\pi i/p} \text{ is a } p\text{-root of unity.} \quad (3)$$

- Each $\zeta^\xi, \xi \neq 0$, is a primitive $p$-th root of unity itself.
- All powers $(\zeta^\xi)^s$ are distinct, so $\hat{1}_E(\xi)$ is a subset sum of all primitive $p$-th roots of unity $(\xi \neq 0)$. 

The polynomial $1 + x + x^2 + \cdots + x^{p-1} - 1$ is the minimal polynomial over $\mathbb{Q}$ of each primitive root of unity (there are $p-1$ of them). It divides any polynomial in $\mathbb{Q}[x]$ which vanishes on some primitive $p$-th root of unity. The only subset sums of all roots of unity which vanish are the empty and the full sum ($E = \emptyset$ or $E = \mathbb{Z}_p$). So in $\mathbb{Z}_p$ the triple correlation $N_E(\cdot, \cdot)$ determines $E$ up to translation.
Suppose $n = p$ is a prime, $E \subseteq \mathbb{Z}_p$. Then

$$\hat{1}_E(\xi) = \frac{1}{p} \sum_{s \in E} (\zeta^s) \xi, \quad \zeta = e^{-2\pi i/p} \text{ is a } p\text{-root of unity.} \quad (3)$$

- Each $\zeta^\xi, \xi \neq 0$, is a primitive $p$-th root of unity itself.
- All powers $(\zeta^\xi)^s$ are distinct, so $\hat{1}_E(\xi)$ is a subset sum of all primitive $p$-th roots of unity ($\xi \neq 0$).
- The polynomial $1 + x + x^2 + \cdots + x^{p-1}$ is the *minimal polynomial* over $\mathbb{Q}$ of each primitive root of unity (there are $p - 1$ of them).
Suppose \( n = p \) is a prime, \( E \subseteq \mathbb{Z}_p \). Then

\[
\hat{1}_E(\xi) = \frac{1}{p} \sum_{s \in E} (\zeta^\xi)^s, \quad \zeta = e^{-2\pi i/p} \text{ is a } p\text{-root of unity.}
\]

(3)

- Each \( \zeta^\xi, \xi \neq 0 \), is a primitive \( p\)-th root of unity itself.
- All powers \( (\zeta^\xi)^s \) are distinct, so \( \hat{1}_E(\xi) \) is a subset sum of all primitive \( p\)-th roots of unity \( (\xi \neq 0) \).
- The polynomial \( 1 + x + x^2 + \cdots + x^{p-1} \) is the \textit{minimal polynomial} over \( \mathbb{Q} \) of each primitive root of unity \( (\text{there are } p - 1 \text{ of them}) \).
- It divides any polynomial in \( \mathbb{Q}[x] \) which vanishes on some primitive \( p\)-th root of unity.
Triple correlations in $\mathbb{Z}_p$: an application (conclusion)

- Suppose $n = p$ is a prime, $E \subseteq \mathbb{Z}_p$. Then

$$\hat{1}_E(\xi) = \frac{1}{p} \sum_{s \in E} (\zeta^s\xi)^s, \quad \zeta = e^{-2\pi i/p} \text{ is a } p\text{-root of unity.} \quad (3)$$

- Each $\zeta^\xi, \xi \neq 0$, is a primitive $p$-th root of unity itself.
- All powers $(\zeta^\xi)^s$ are distinct, so $\hat{1}_E(\xi)$ is a subset sum of all primitive $p$-th roots of unity ($\xi \neq 0$).
- The polynomial $1 + x + x^2 + \cdots + x^{p-1}$ is the minimal polynomial over $\mathbb{Q}$ of each primitive root of unity (there are $p - 1$ of them).
- It divides any polynomial in $\mathbb{Q}[x]$ which vanishes on some primitive $p$-th root of unity.
- The only subset sums of all roots of unity which vanish are the empty and the full sum ($E = \emptyset$ or $E = \mathbb{Z}_p$).
Triple correlations in $\mathbb{Z}_p$: an application (conclusion)

- Suppose $n = p$ is a prime, $E \subseteq \mathbb{Z}_p$. Then

$$\hat{1}_E(\xi) = \frac{1}{p} \sum_{s \in E} (\zeta^\xi)^s, \quad \zeta = e^{-2\pi i/p} \text{ is a } p\text{-root of unity.} \quad (3)$$

- Each $\zeta^\xi, \xi \neq 0$, is a primitive $p$-th root of unity itself.
- All powers $(\zeta^\xi)^s$ are distinct, so $\hat{1}_E(\xi)$ is a subset sum of all primitive $p$-th roots of unity ($\xi \neq 0$).
- The polynomial $1 + x + x^2 + \cdots + x^{p-1}$ is the *minimal polynomial* over $\mathbb{Q}$ of each primitive root of unity (there are $p - 1$ of them).
- It divides any polynomial in $\mathbb{Q}[x]$ which vanishes on some primitive $p$-th root of unity
- The only subset sums of all roots of unity which vanish are the empty and the full sum ($E = \mathbb{Z}_p$).
- So in $\mathbb{Z}_p$ the triple correlation $N_E(\cdot, \cdot)$ determines $E$ up to translation.
1 ≤ p ≤ q ⇐⇒ \( L^q(\mathbb{T}) \subseteq L^p(\mathbb{T}) \): nested \( L^p \) spaces. True on compact groups.
The basics of the FT on the torus (circle) $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$

- $1 \leq p \leq q \iff L^q(\mathbb{T}) \subseteq L^p(\mathbb{T})$: nested $L^p$ spaces. True on compact groups.
- $f \in L^1(\mathbb{T})$: we write $f(x) \sim \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{2\pi ikx}$ to denote the Fourier series of $f$. No claim of convergence is made.
The basics of the FT on the torus (circle) $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$

- $1 \leq p \leq q \iff L^q(\mathbb{T}) \subseteq L^p(\mathbb{T})$: nested $L^p$ spaces. True on compact groups.
- $f \in L^1(\mathbb{T})$: we write $f(x) \sim \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{2\pi ikx}$ to denote the Fourier series of $f$. No claim of convergence is made.
- The Fourier coefficients of $f(x) = e^{2\pi ikx}$ is the sequence $\hat{f}(n) = \delta_{k,n}$. 

The Fourier series of a trig. poly. $f(x) = \sum_{N} a_k e^{2\pi ikx}$ is the sequence $\ldots, a_{-N}, 0, a_{-N+1}, \ldots, a_0, \ldots, a_N, 0, \ldots$.

Symmetric partial sums of the Fourier series of $f$: $S_N(f; x) = \sum_{N} \hat{f}(k)e^{2\pi ikx}$.

From $\hat{f}^*g = \hat{f} \cdot \hat{g}$ we get easily $S_N(f; x) = f(x) \ast D_N(x)$, where $D_N(x) = \frac{N}{\pi} \sum_{k=-N}^{N} e^{2\pi ikx} = \sin \frac{2\pi (N+1)}{2}x \sin \pi x$ (Dirichlet kernel of order $N$).
The basics of the FT on the torus (circle) $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$

- $1 \leq p \leq q \iff L^q(\mathbb{T}) \subseteq L^p(\mathbb{T})$: nested $L^p$ spaces. True on compact groups.
- $f \in L^1(\mathbb{T})$: we write $f(x) \sim \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{2\pi ikx}$ to denote the Fourier series of $f$. No claim of convergence is made.
- The Fourier coefficients of $f(x) = e^{2\pi ikx}$ is the sequence $\hat{f}(n) = \delta_{k,n}$.
- The Fourier series of a trig. poly. $f(x) = \sum_{k=-N}^{N} a_k e^{2\pi ikx}$ is the sequence $\ldots, 0, 0, a_{-N}, a_{-N+1}, \ldots, a_0, \ldots, a_N, 0, 0, \ldots$. 

The basics of the FT on the torus (circle) \( \mathbb{T} = \mathbb{R}/2\pi \mathbb{Z} \)

- \( 1 \leq p \leq q \iff L^q(\mathbb{T}) \subseteq L^p(\mathbb{T}) \): nested \( L^p \) spaces. True on compact groups.
- \( f \in L^1(\mathbb{T}) \): we write \( f(x) \sim \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{2\pi ikx} \) to denote the Fourier series of \( f \). No claim of convergence is made.
- The Fourier coefficients of \( f(x) = e^{2\pi ikx} \) is the sequence \( \hat{f}(n) = \delta_{k,n} \).
- The Fourier series of a trig. poly. \( f(x) = \sum_{k=-N}^{N} a_k e^{2\pi ikx} \) is the sequence \( \ldots, 0, 0, a_{-N}, a_{-N+1}, \ldots, a_0, \ldots, a_N, 0, 0, \ldots \).
- Symmetric partial sums of the Fourier series of \( f \):
  \[ S_N(f; x) = \sum_{k=-N}^{N} \hat{f}(k)e^{2\pi ikx} \]
The basics of the FT on the torus (circle) $\mathbb{T} = \mathbb{R}/2\pi \mathbb{Z}$

- $1 \leq p \leq q \iff L^q(\mathbb{T}) \subseteq L^p(\mathbb{T})$: nested $L^p$ spaces. True on compact groups.
- $f \in L^1(\mathbb{T})$: we write $f(x) \sim \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{2\pi ikx}$ to denote the Fourier series of $f$. No claim of convergence is made.
- The Fourier coefficients of $f(x) = e^{2\pi ikx}$ is the sequence $\hat{f}(n) = \delta_{k,n}$.
- The Fourier series of a trig. poly. $f(x) = \sum_{k=-N}^{N} a_k e^{2\pi ikx}$ is the sequence $\ldots, 0, 0, a_{-N}, a_{-N+1}, \ldots, a_0, \ldots, a_N, 0, 0, \ldots$.
- Symmetric partial sums of the Fourier series of $f$: $S_N(f; x) = \sum_{k=-N}^{N} \hat{f}(k)e^{2\pi ikx}$
- From $\hat{f} * \hat{g} = \hat{f} \cdot \hat{g}$ we get easily $S_N(f; x) = f(x) * D_N(x)$, where

$$D_N(x) = \sum_{k=-N}^{N} e^{2\pi ikx} = \frac{\sin 2\pi (N + \frac{1}{2})x}{\sin \pi x} \quad (\text{Dirichlet kernel of order } N)$$
The Dirichlet kernel $D_N(x)$ for $N = 10$
**Pointwise convergence**

- **Important:** $\|D_N\|_1 \geq C \log N$, as $N \to \infty$
Pointwise convergence

- Important: \( \|D_N\|_1 \geq C \log N \), as \( N \to \infty \)

- \( T_N : f \to S_N(f;x) = D_N \ast f(x) \) is a (continuous) linear functional \( C(\mathbb{T}) \to \mathbb{C} \). From the inequality \( \|D_N \ast f\|_\infty \leq \|D_N\|_1 \|f\|_\infty \)
Important: $\|D_N\|_1 \geq C \log N$, as $N \to \infty$

$T_N : f \rightarrow S_N(f; x) = D_N \ast f(x)$ is a (continuous) linear functional $C(\mathbb{T}) \rightarrow \mathbb{C}$. From the inequality $\|D_N \ast f\|_\infty \leq \|D_N\|_1 \|f\|_\infty$

$\|T_N\| = \|D_N\|_1$ is unbounded
Important: $\|D_N\|_1 \geq C \log N$, as $N \to \infty$

$T_N : f \to S_N(f;x) = D_N \ast f(x)$ is a (continuous) linear functional $C(\mathbb{T}) \to \mathbb{C}$. From the inequality $\|D_N \ast f\|_\infty \leq \|D_N\|_1 \|f\|_\infty$

$\|T_N\| = \|D_N\|_1$ is unbounded

Banach-Steinhaus (uniform boundedness principle) $\implies$
Given $x$ there are many continuous functions $f$ such that $T_N(f)$ is unbounded
Important:  \[ \|D_N\|_1 \geq C \log N, \text{ as } N \to \infty \]

\[ T_N : f \to S_N(f; x) = D_N * f(x) \] is a (continuous) linear functional \( C(\mathbb{T}) \to \mathbb{C} \). From the inequality \[ \|D_N * f\|_\infty \leq \|D_N\|_1 \|f\|_\infty \]

\[ \|T_N\| = \|D_N\|_1 \text{ is unbounded} \]

**Banach-Steinhaus** (uniform boundedness principle) \[ \implies \]
Given \( x \) there are many continuous functions \( f \) such that \( T_N(f) \) is unbounded

Consequence: In general \( S_N(f; x) \) does not converge pointwise to \( f(x) \), even for continuous \( f \)
Look at the arithmetical means of $S_N(f; x)$

$$
\sigma_N(f; x) = \frac{1}{N + 1} \sum_{n=0}^{N} S_n(f; x) = K_N \ast f(x)
$$
Summability

- Look at the arithmetical means of \( S_N(f; x) \)

\[
\sigma_N(f; x) = \frac{1}{N+1} \sum_{n=0}^{N} S_n(f; x) = K_N * f(x)
\]

- The **Fejér** kernel \( K_N(x) \) is the mean of the **Dirichlet** kernels

\[
K_N(x) = \sum_{n=-N}^{N} \left( 1 - \frac{|n|}{N+1} \right) e^{2\pi inx} = \frac{1}{N+1} \left( \frac{\sin \pi(N+1)x}{\sin \pi x} \right)^2 \geq 0.
\]
Summability

- Look at the arithmetical means of $S_N(f; x)$

\[ \sigma_N(f; x) = \frac{1}{N + 1} \sum_{n=0}^{N} S_n(f; x) = K_N \ast f(x) \]

- The Fejér kernel $K_N(x)$ is the mean of the Dirichlet kernels

\[ K_N(x) = \sum_{n=-N}^{N} \left( 1 - \frac{|n|}{N + 1} \right) e^{2\pi i n x} = \frac{1}{N + 1} \left( \frac{\sin \pi (N + 1)x}{\sin \pi x} \right)^2 \geq 0. \]

- $K_N(x)$ is an approximate identity:
  
  (a) $\int_{\mathbb{T}} K_N(x) \, dx = \hat{K}_N(0) = 1$,
  
  (b) $\|K_N\|_1$ is bounded ($\|K_N\|_1 = 1$, from nonnegativity and (a)),
  
  (c) for any $\epsilon > 0$ we have $\int_{|x| > \epsilon} |K_N(x)| \, dx \to 0$, as $N \to \infty$
The Fejér kernel $D_N(x)$ for $N = 10$
Summability (continued)

- $K_N$ approximate identity $\Rightarrow K_N * f(x) \to f(x)$, in some Banach spaces. These can be:

- $C(T)$ normed with $\|\cdot\|_{\infty}$: If $f \in C(T)$ then $\sigma_N(f; x) \to f(x)$ uniformly in $T$.

- $L_p(T), 1 \leq p < \infty$: If $f \in L_p(T)$ then $\|\sigma_N(f; x) - f(x)\|_p \to 0$.

- $C^n(T)$, all $n$-times $C$-differentiable functions, normed with $\|f\|_{C^n} = \sum_{k=0}^{n} \|f^{(k)}\|_{\infty}$.

Summability implies uniqueness: the Fourier series of $f \in L_1(T)$ determines the function.

Another consequence: trig. polynomials are dense in $L_p(T), C(T), C^n(T)$.

Another important summability kernel: the Poisson kernel $P(r, x) = \sum_{k \in \mathbb{Z}} r^k e^{2\pi ikx}$, $0 < r < 1$: absolute convergence obvious.

Significant for the theory of analytic functions.
Summability (continued)

- $K_N$ approximate identity \( \Rightarrow K_N \ast f(x) \to f(x) \), in some Banach spaces. These can be:
- $C(\mathbb{T})$ normed with $\| \cdot \|_\infty$: If $f \in C(\mathbb{T})$ then $\sigma_N(f; x) \to f(x)$ uniformly in $\mathbb{T}$.

Significant for the theory of analytic functions.

Mihalis Kolountzakis (U. of Crete)
Summability (continued)

- $K_N$ approximate identity $\implies K_N * f(x) \to f(x)$, in some Banach spaces. These can be:
- $C(\mathbb{T})$ normed with $\|\cdot\|_\infty$: If $f \in C(\mathbb{T})$ then $\sigma_N(f; x) \to f(x)$ uniformly in $\mathbb{T}$.
- $L^p(\mathbb{T})$, $1 \leq p < \infty$: If $f \in L^p(\mathbb{T})$ then $\|\sigma_N(f; x) - f(x)\|_p \to 0$.
Summability (continued)

- $K_N$ approximate identity $\implies K_N \ast f(x) \rightarrow f(x)$, in some Banach spaces. These can be:
  - $C(\mathbb{T})$ normed with $\| \cdot \|_\infty$: If $f \in C(\mathbb{T})$ then $\sigma_N(f; x) \rightarrow f(x)$ uniformly in $\mathbb{T}$.
  - $L^p(\mathbb{T})$, $1 \leq p < \infty$: If $f \in L^p(\mathbb{T})$ then $\| \sigma_N(f; x) - f(x) \|_p \rightarrow 0$
  - $C^n(\mathbb{T})$, all $n$-times $C$-differentiable functions, normed with $\| f \|_{C^n} = \sum_{k=0}^{n} \| f^{(k)} \|_\infty$
Summability (continued)

- $K_N$ approximate identity $\implies K_N * f(x) \to f(x)$, in some Banach spaces. These can be:
- $C(\mathbb{T})$ normed with $\| \cdot \|_\infty$: If $f \in C(\mathbb{T})$ then $\sigma_N(f; x) \to f(x)$ uniformly in $\mathbb{T}$.
- $L^p(\mathbb{T})$, $1 \leq p < \infty$: If $f \in L^p(\mathbb{T})$ then $\|\sigma_N(f; x) - f(x)\|_p \to 0$
- $C^n(\mathbb{T})$, all $n$-times $C$-differentiable functions, normed with $\|f\|_{C^n} = \sum_{k=0}^{n} \|f^{(k)}\|_\infty$
- Summability implies uniqueness: the Fourier series of $f \in L^1(\mathbb{T})$ determines the function.
Summability (continued)

- $K_N$ approximate identity $\implies K_N \ast f(x) \to f(x)$, in some Banach spaces. These can be:

- $C(\mathbb{T})$ normed with $\|\cdot\|_{\infty}$: If $f \in C(\mathbb{T})$ then $\sigma_N(f; x) \to f(x)$ uniformly in $\mathbb{T}$.

- $L^p(\mathbb{T})$, $1 \leq p < \infty$: If $f \in L^p(\mathbb{T})$ then $\|\sigma_N(f; x) - f(x)\|_p \to 0$

- $C^n(\mathbb{T})$, all $n$-times $C$-differentiable functions, normed with $\|f\|_{C^n} = \sum_{k=0}^n \|f^{(k)}\|_{\infty}$

- Summability implies uniqueness: the Fourier series of $f \in L^1(\mathbb{T})$ determines the function.

- Another consequence: trig. polynomials are dense in $L^p(\mathbb{T})$, $C(\mathbb{T})$, $C^n(\mathbb{T})$
Summability (continued)

- $K_N$ approximate identity $\implies K_N \ast f(x) \to f(x)$, in some Banach spaces. These can be:
- $C(\mathbb{T})$ normed with $\|\cdot\|_\infty$: If $f \in C(\mathbb{T})$ then $\sigma_N(f; x) \to f(x)$ uniformly in $\mathbb{T}$.
- $L^p(\mathbb{T})$, $1 \leq p < \infty$: If $f \in L^p(\mathbb{T})$ then $\|\sigma_N(f; x) - f(x)\|_p \to 0$.
- $C^n(\mathbb{T})$, all $n$-times $C$-differentiable functions, normed with $\|f\|_{C^n} = \sum_{k=0}^n \|f^{(k)}\|_\infty$.
- Summability implies uniqueness: the Fourier series of $f \in L^1(\mathbb{T})$ determines the function.
- Another consequence: trig. polynomials are dense in $L^p(\mathbb{T})$, $C(\mathbb{T})$, $C^n(\mathbb{T})$.
- Another important summability kernel: the Poisson kernel

$$P(r, x) = \sum_{k \in \mathbb{Z}} r^k e^{2\pi i k x}, \quad 0 < r < 1: \text{absolute convergence obvious}$$

Significant for the theory of analytic functions.
The decay of the Fourier coefficients at $\infty$

- Obvious: $\hat{f}(n) \leq \|f\|_1$
The decay of the Fourier coefficients at $\infty$

- Obvious: $\hat{f}(n) \leq \|f\|_1$
- **Riemann-Lebesgue Lemma**: $\lim_{|n| \to \infty} \hat{f}(n) = 0$ if $f \in L^1(\mathbb{T})$. Obviously true for trig. polynomials and they are dense in $L^1(\mathbb{T})$. 

$\hat{f}(x) = \int_{-\infty}^{\infty} g(t) \, dt$, where $\int g(t) \, dt = 0$:

$f$ is an integral $\Rightarrow \hat{f}(n) = o(1/n)$: the "smoother" $f$ is the better decay for the FT of $f$.

Another condition that imposes "decay": $f \in L^2(\mathbb{T})$ $\Rightarrow$ absolute convergence for the Fourier Series of $f$. 

$\sum_{n > 0} |\hat{f}(n)|^2 < \infty$.

$\sum_{n \neq 0} \hat{f}(n) \sin nx \log n$ is not a Fourier series.
The decay of the Fourier coefficients at $\infty$

- **Obvious:** $\hat{f}(n) \leq \|f\|_1$

- **Riemann-Lebesgue Lemma:** $\lim_{|n| \to \infty} \hat{f}(n) = 0$ if $f \in L^1(\mathbb{T})$. Obviously true for trig. polynomials and they are dense in $L^1(\mathbb{T})$.

- Can go to 0 arbitrarily slowly if we only assume $f \in L^1$. 

\[ f(x) = \int_0^x g(t) \, dt, \text{ where } \int g = 0: \hat{f}(n) = \frac{1}{2\pi} \hat{g}(n) (\text{Fubini}) \]

Previous implies:

\[ \hat{f}(|n|) = -\hat{f}(-|n|) \geq 0 \Rightarrow \sum_{n \neq 0} \frac{\hat{f}(n)}{n} < \infty. \]

$\sum_{n > 0} \sin nx \log n$ is not a Fourier series.

$f$ is an integral $\Rightarrow \hat{f}(n) = o\left(\frac{1}{n}\right)$: the "smoother" $f$ is the better decay for the FT of $f$.

$f \in C^2(\mathbb{T}) = \Rightarrow$ absolute convergence for the Fourier Series of $f$. 

Another condition that imposes "decay": $f \in L^2(\mathbb{T}) = \Rightarrow \sum_{n} |\hat{f}(n)|^2 < \infty$. 
The decay of the Fourier coefficients at $\infty$

- Obvious: $\hat{f}(n) \leq \| f \|_1$

- **Riemann-Lebesgue Lemma**: $\lim_{|n| \to \infty} \hat{f}(n) = 0$ if $f \in L^1(\mathbb{T})$.
  Obviously true for trig. polynomials and they are dense in $L^1(\mathbb{T})$.

- Can go to 0 arbitrarily slowly if we only assume $f \in L^1$.

- $f(x) = \int_0^x g(t) \, dt$, where $\int g = 0$: $\hat{f}(n) = \frac{1}{2\pi in} \hat{g}(n)$ (Fubini)
The decay of the Fourier coefficients at $\infty$

- Obvious: $\hat{f}(n) \leq \|f\|_1$

- **Riemann-Lebesgue Lemma**: $\lim_{|n| \to \infty} \hat{f}(n) = 0$ if $f \in L^1(\mathbb{T})$.
  Obviously true for trig. polynomials and they are dense in $L^1(\mathbb{T})$.

- Can go to 0 arbitrarily slowly if we only assume $f \in L^1$.

- $f(x) = \int_0^x g(t) \, dt$, where $\int g = 0$: $\hat{f}(n) = \frac{1}{2\pi i n} \hat{g}(n)$ (Fubini)

- Previous implies: $\hat{f}(|n|) = -\hat{f}(-|n|) \geq 0 \implies \sum_{n \neq 0} \hat{f}(n)/n < \infty$. 

The decay of the Fourier coefficients at $\infty$

- **Obvious:** $\hat{f}(n) \leq \|f\|_1$
- **Riemann-Lebesgue Lemma:** $\lim_{|n| \to \infty} \hat{f}(n) = 0$ if $f \in L^1(\mathbb{T})$. Obviously true for trig. polynomials and they are dense in $L^1(\mathbb{T})$.
- Can go to 0 arbitrarily slowly if we only assume $f \in L^1$.
- $f(x) = \int_0^x g(t) \, dt$, where $\int g = 0$: $\hat{f}(n) = \frac{1}{2\pi i n} \hat{g}(n)$ (Fubini)
- Previous implies: $\hat{f}(|n|) = -\hat{f}(-|n|) \geq 0 \implies \sum_{n \neq 0} \hat{f}(n)/n < \infty$.
- $\sum_{n > 0} \frac{\sin nx}{\log n}$ is not a Fourier series.
The decay of the Fourier coefficients at $\infty$

- **Obvious:** $\hat{f}(n) \leq \|f\|_1$
- **Riemann-Lebesgue Lemma:** $\lim_{|n| \to \infty} \hat{f}(n) = 0$ if $f \in L^1(\mathbb{T})$. Obviously true for trig. polynomials and they are dense in $L^1(\mathbb{T})$.
- Can go to 0 arbitrarily slowly if we only assume $f \in L^1$.
- $f(x) = \int_0^x g(t) \, dt$, where $\int g = 0$: $\hat{f}(n) = \frac{1}{2\pi i n} \hat{g}(n)$ (Fubini)
- Previous implies: $\hat{f}(|n|) = -\hat{f}(-|n|) \geq 0 \implies \sum_{n \neq 0} \hat{f}(n) / n < \infty$.
- $\sum_{n > 0} \frac{\sin nx}{\log n}$ is not a Fourier series.
- $f$ is an integral $\implies \hat{f}(n) = o(1/n)$: the "smoother" $f$ is the better decay for the FT of $f$
The decay of the Fourier coefficients at $\infty$

- **Obvious:** $\hat{f}(n) \leq \|f\|_1$
- **Riemann-Lebesgue Lemma:** $\lim_{|n| \to \infty} \hat{f}(n) = 0$ if $f \in L^1(\mathbb{T})$.
  Obviously true for trig. polynomials and they are dense in $L^1(\mathbb{T})$.
- Can go to 0 arbitrarily slowly if we only assume $f \in L^1$.
- $f(x) = \int_0^x g(t) \, dt$, where $\int g = 0$: $\hat{f}(n) = \frac{1}{2\pi i n} \hat{g}(n)$ (Fubini)
- Previous implies: $\hat{f}(|n|) = -\hat{f}(-|n|) \geq 0 \implies \sum_{n \neq 0} \hat{f}(n)/n < \infty$.
- $\sum_{n > 0} \frac{\sin nx}{\log n}$ is not a Fourier series.
- $f$ is an integral $\implies \hat{f}(n) = o(1/n)$: the “smoother” $f$ is the better decay for the FT of $f$.
- $f \in C^2(\mathbb{T}) \implies$ absolute convergence for the Fourier Series of $f$. 
The decay of the Fourier coefficients at $\infty$

- **Obvious:** $\hat{f}(n) \leq \|f\|_1$
- **Riemann-Lebesgue Lemma:** $\lim_{|n| \to \infty} \hat{f}(n) = 0$ if $f \in L^1(\mathbb{T})$.
  Obviously true for trig. polynomials and they are dense in $L^1(\mathbb{T})$.
- Can go to 0 arbitrarily slowly if we only assume $f \in L^1$.
- $f(x) = \int_0^x g(t) \, dt$, where $\int g = 0$: $\hat{f}(n) = \frac{1}{2\pi in} \hat{g}(n)$ (Fubini)
- Previous implies: $\hat{f}(|n|) = -\hat{f}(-|n|) \geq 0 \implies \sum_{n \neq 0} \hat{f}(n)/n < \infty$.
- $\sum_{n>0} \frac{\sin nx}{\log n}$ is not a Fourier series.
- $f$ is an integral $\implies \hat{f}(n) = o(1/n)$: the “smoother” $f$ is the better decay for the FT of $f$.
- $f \in C^2(\mathbb{T}) \implies$ absolute convergence for the Fourier Series of $f$.
- Another condition that imposes “decay”:
  $f \in L^2(\mathbb{T}) \implies \sum_n \left|\hat{f}(n)\right|^2 < \infty$. 
Interpolation of operators

- $T$ is bounded linear operator on dense subsets of $L^{p_1}$ and $L^{p_2}$:

$$\|Tf\|_{q_1} \leq C_1 \|f\|_{p_1}, \quad \|Tf\|_{q_2} \leq C_2 \|f\|_{p_2}$$
Interpolation of operators

- $T$ is bounded linear operator on dense subsets of $L^{p_1}$ and $L^{p_2}$:

  $\|Tf\|_{q_1} \leq C_1\|f\|_{p_1}$, $\|Tf\|_{q_2} \leq C_2\|f\|_{p_2}$

- **Riesz-Thorin** interpolation theorem: $T : L^p \to L^q$ for any $p$ between $p_1, p_2$ (all $p$’s and $q$’s $\geq 1$).
Interpolation of operators

- $T$ is bounded linear operator on dense subsets of $L^{p_1}$ and $L^{p_2}$:

  $$\|Tf\|_{q_1} \leq C_1\|f\|_{p_1}, \quad \|Tf\|_{q_2} \leq C_2\|f\|_{p_2}$$

- **Riesz-Thorin interpolation theorem**: $T : L^p \to L^q$ for any $p$ between $p_1, p_2$ (all $p$’s and $q$’s $\geq 1$).

- $p$ and $q$ are related by:

  $$\frac{1}{p} = t\frac{1}{p_1} + (1-t)\frac{1}{p_2} \quad \Rightarrow \quad \frac{1}{q} = t\frac{1}{q_1} + (1-t)\frac{1}{q_2}$$
Interpolation of operators

- $T$ is bounded linear operator on dense subsets of $L^{p_1}$ and $L^{p_2}$:
  \[ \| Tf \|_{q_1} \leq C_1 \| f \|_{p_1}, \quad \| Tf \|_{q_2} \leq C_2 \| f \|_{p_2} \]

- **Riesz-Thorin** interpolation theorem: $T : L^p \to L^q$ for any $p$ between $p_1, p_2$ (all $p$’s and $q$’s $\geq 1$).
  - $p$ and $q$ are related by:
    \[
    \frac{1}{p} = t \frac{1}{p_1} + (1 - t) \frac{1}{p_2} \quad \Rightarrow \quad \frac{1}{q} = t \frac{1}{q_1} + (1 - t) \frac{1}{q_2}
    \]
  - $\| T \|_{L^p \to L^q} \leq C_1^t C_2^{(1-t)}$
Interpolation of operators

- $T$ is bounded linear operator on dense subsets of $L^{p_1}$ and $L^{p_2}$:
  \[ \| Tf \|_{q_1} \leq C_1 \| f \|_{p_1}, \quad \| Tf \|_{q_2} \leq C_2 \| f \|_{p_2} \]

- **Riesz-Thorin interpolation theorem**: $T : L^p \to L^q$ for any $p$ between $p_1, p_2$ (all $p$’s and $q$’s $\geq 1$).

- $p$ and $q$ are related by:
  \[
  \frac{1}{p} = t \frac{1}{p_1} + (1 - t) \frac{1}{p_2} \quad \Rightarrow \quad \frac{1}{q} = t \frac{1}{q_1} + (1 - t) \frac{1}{q_2}
  \]

- $\| T \|_{L^p \to L^q} \leq C_1^t C_2^{1-t}$

- The exponents $p, q, \ldots$ are allowed to be $\infty$. 
Interpolation of operators: the $1/p, 1/q$ plane

The diagram illustrates the interpolation of operators in the $1/p, 1/q$ plane. Points $(1/p_1, 1/q_1)$, $(1/p, 1/q)$, and $(1/p_2, 1/q_2)$ are marked on the plane, indicating the interpolation points between different spaces.
The Hausdorff-Young inequality

**Hausdorff-Young:** Suppose $1 \leq p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, and $f \in L^p(\mathbb{T})$. It follows that

$$\left\| \hat{f} \right\|_{L^q(\mathbb{Z})} \leq C_p \left\| f \right\|_{L^p(\mathbb{T})}$$
**Hausdorff-Young:** Suppose $1 \leq p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, and $f \in L^p(\mathbb{T})$. It follows that

$$\left\| \hat{f} \right\|_{L^q(\mathbb{Z})} \leq C_p \left\| f \right\|_{L^p(\mathbb{T})}$$

- False if $p > 2$. 

Clearly true if $p = 1$ (trivial) or $p = 2$ (Parseval). Use Riesz-Thorin interpolation for $1 < p < 2$ for the operator $f \rightarrow \hat{f}$ from $L^p(\mathbb{T}) \rightarrow L^q(\mathbb{Z})$. 
The Hausdorff-Young inequality

- **Hausdorff-Young**: Suppose $1 \leq p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, and $f \in L^p(\mathbb{T})$. It follows that

$$\|\hat{f}\|_{L^q(\mathbb{Z})} \leq C_p \|f\|_{L^p(\mathbb{T})}$$

- False if $p > 2$.
- Clearly true if $p = 1$ (trivial) or $p = 2$ (Parseval).
The Hausdorff-Young inequality

- **Hausdorff-Young**: Suppose $1 \leq p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, and $f \in L^p(\mathbb{T})$. It follows that

$$\|\hat{f}\|_{L^q(\mathbb{Z})} \leq C_p \|f\|_{L^p(\mathbb{T})}$$

- False if $p > 2$.
- Clearly true if $p = 1$ (trivial) or $p = 2$ (Parseval).
- Use **Riesz-Thorin** interpolation for $1 < p < 2$ for the operator $f \rightarrow \hat{f}$ from $L^p(\mathbb{T}) \rightarrow L^q(\mathbb{Z})$. 
An application: the isoperimetric inequality

Suppose $\Gamma$ is a simple closed curve in the plane with perimeter $L$ enclosing area $A$.

$$A \leq \frac{1}{4\pi} L^2 \quad \text{(isoperimetric inequality)}$$

Equality holds only when $\Gamma$ is a circle.
Suppose $\Gamma$ is a simple closed curve in the plane with perimeter $L$ enclosing area $A$.

\[
A \leq \frac{1}{4\pi} L^2 \quad \text{(isoperimetric inequality)}
\]

Equality holds only when $\Gamma$ is a circle.

Wirtinger’s inequality: if $f \in C^\infty(\mathbb{T})$ then

\[
\int_0^1 \left| f(x) - \hat{f}(0) \right|^2 dx \leq \frac{1}{4\pi^2} \int_0^1 |f'(x)|^2 dx.
\] (4)
An application: the isoperimetric inequality

- Suppose $\Gamma$ is a simple closed curve in the plane with perimeter $L$ enclosing area $A$.

  $$ A \leq \frac{1}{4\pi} L^2 \quad \text{(isoperimetric inequality)} $$

  Equality holds only when $\Gamma$ is a circle.

- Wirtinger’s inequality: if $f \in C^\infty(T)$ then

  $$ \int_0^1 \left| f(x) - \hat{f}(0) \right|^2 \, dx \leq \frac{1}{4\pi^2} \int_0^1 |f'(x)|^2 \, dx. \quad (4) $$

- By smoothness $f(x)$ equals its Fourier series and so does $f'(x) = 2\pi i \sum_n nf(n)e^{2\pi inx}$
An application: the isoperimetric inequality

- Suppose $\Gamma$ is a simple closed curve in the plane with perimeter $L$ enclosing area $A$.

$$A \leq \frac{1}{4\pi}L^2 \quad \text{(isoperimetric inequality)}$$

Equality holds only when $\Gamma$ is a circle.

- Wirtinger’s inequality: if $f \in C^\infty(\mathbb{T})$ then

$$\int_0^1 \left| f(x) - \hat{f}(0) \right|^2 \, dx \leq \frac{1}{4\pi^2} \int_0^1 \left| f'(x) \right|^2 \, dx. \quad (4)$$

- By smoothness $f(x)$ equals its Fourier series and so does $f'(x) = 2\pi i \sum_n n\hat{f}(n)e^{2\pi inx}$

- FT is an isometry (Parseval) so LHS of (4) is $\sum_{n\neq 0} \left| \hat{f}(n) \right|^2$ while the RHS is $\sum_{n\neq 0} n\left| \hat{f}(n) \right|^2$ so (4) holds.
An application: the isoperimetric inequality

- Suppose $\Gamma$ is a simple closed curve in the plane with perimeter $L$ enclosing area $A$.

  \[ A \leq \frac{1}{4\pi} L^2 \]  
  \text{(isoperimetric inequality)}

  Equality holds only when $\Gamma$ is a circle.

- Wirtinger’s inequality: if $f \in C^\infty(\mathbb{T})$ then

  \[ \int_0^1 \left| f(x) - \hat{f}(0) \right|^2 \, dx \leq \frac{1}{4\pi^2} \int_0^1 \left| f'(x) \right|^2 \, dx. \]  
  \text{(4)}

- By smoothness $f(x)$ equals its Fourier series and so does $f'(x) = 2\pi i \sum_n n f(n) e^{2\pi i nx}$.

- FT is an isometry (Parseval) so LHS of (4) is $\sum_{n\neq 0} \left| \hat{f}(n) \right|^2$ while the RHS is $\sum_{n\neq 0} n \left| \hat{f}(n) \right|^2$ so (4) holds.

- Equality in (4) precisely when $f(x) = \hat{f}(-1)e^{-2\pi ix} + \hat{f}(0) + \hat{f}(1)e^{2\pi ix}$.  

Mihalis Kolountzakis (U. of Crete)
An application: the isoperimetric inequality (continued)

- Hurwitz’ proof. First assume $\Gamma$ is smooth, has $L = 1$. 

\[ \text{Green's Theorem } \Rightarrow \text{area } A = \int_0^1 x(s) y'(s) \, ds: \]
\[ A = \int (x(s) - \hat{x}(0)) y'(s) = \frac{1}{4\pi} \int (2\pi(x(s) - \hat{x}(0)))^2 + y'(s)^2 \leq \frac{1}{4\pi} \int x'(s)^2 + y'(s)^2 \quad \text{Wirtinger's ineq}) \]

For equality must have $x(s) = a \cos 2\pi s + b \sin 2\pi s + c$, $y'(s) = 2\pi(x(s) - \hat{x}(0))$. So $x(s)^2 + y(s)^2$ constant if $c = 0$. 

Hurwitz’ proof. First assume $\Gamma$ is smooth, has $L = 1$. 

Parametrization of $\Gamma$: $(x(s), y(s))$, $0 \leq s \leq 1$ w.r.t. arc length $s$. 
An application: the isoperimetric inequality (continued)

- Hurwitz’ proof. First assume $\Gamma$ is smooth, has $L = 1$.
- Parametrization of $\Gamma$: $(x(s), y(s))$, $0 \leq s \leq 1$ w.r.t. arc length $s$
- $x, y \in C^\infty(\mathbb{T})$, $(x'(s))^2 + (y'(s))^2 = 1$. 

\[ \text{Green's Theorem } \Rightarrow \text{area } A = \int_0^1 x(s)y'(s) \, ds : \]
\[ A = \int (x(s) - \hat{x}(0))y'(s) = 1/4\pi \int (2\pi(x(s) - \hat{x}(0)))^2 + y'(s)^2 - (2\pi(x(s) - \hat{x}(0)) - y'(s))^2 \leq 1/4\pi \int x'(s)^2 + y'(s)^2 \quad (\text{Wirtinger's ineq}) \]

For equality must have $x(s) = a \cos 2\pi s + b \sin 2\pi s + c$, $y'(s) = 2\pi(x(s) - \hat{x}(0))$. So $x(s)^2 + y(s)^2$ constant if $c = 0$. 

Mihalis Kolountzakis (U. of Crete)
Hurwitz’ proof. First assume $\Gamma$ is smooth, has $L = 1$.

Parametrization of $\Gamma$: $(x(s), y(s))$, $0 \leq s \leq 1$ w.r.t. arc length $s$

$x, y \in C^\infty(\mathbb{T})$, $(x'(s))^2 + (y'(s))^2 = 1$.

Green’s Theorem $\implies$ area $A = \int_0^1 x(s)y'(s) \, ds$:

$$A = \int (x(s) - \hat{x}(0))y'(s) =$$

$$= \frac{1}{4\pi} \int (2\pi(x(s) - \hat{x}(0)))^2 + y'(s)^2 - (2\pi(x(s) - \hat{x}(0)) - y'(s))^2$$

$$\leq 1/4\pi \int 4\pi^2(x(s) - \hat{x}(0))^2 + y'(s)^2 \quad \text{(drop last term)}$$

$$\leq 1/4\pi \int x'(s)^2 + y'(s)^2 \quad (\text{Wirtinger’s ineq})$$

$$= 1/4\pi$$
Hurwitz’ proof. First assume \( \Gamma \) is smooth, has \( L = 1 \).

Parametrization of \( \Gamma \): \((x(s), y(s)), 0 \leq s \leq 1 \) w.r.t. arc length \( s \).

\( x, y \in C^\infty(\mathbb{T}), \quad (x'(s))^2 + (y'(s))^2 = 1. \)

Green’s Theorem \( \implies \) area \( A = \int_0^1 x(s)y'(s) \, ds \):

\[
A = \int (x(s) - \hat{x}(0))y'(s) = \\
= \frac{1}{4\pi} \int (2\pi(x(s) - \hat{x}(0)))^2 + y'(s)^2 - (2\pi(x(s) - \hat{x}(0)) - y'(s))^2
\]

\( \leq \frac{1}{4\pi} \int 4\pi^2(x(s) - \hat{x}(0))^2 + y'(s)^2 \) (drop last term)

\( \leq \frac{1}{4\pi} \int x'(s)^2 + y'(s)^2 \) (Wirtinger’s ineq)

\( = \frac{1}{4\pi} \)

For equality must have \( x(s) = a \cos 2\pi s + b \sin 2\pi s + c, \) \( y'(s) = 2\pi(x(s) - \hat{x}(0)). \) So \( x(s)^2 + y(s)^2 \) constant if \( c = 0. \)
Fourier transform on $\mathbb{R}^n$

- Initially defined only for $f \in L^1(\mathbb{R}^n)$. $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i \xi \cdot x} \, dx$.

Follows: $\|\hat{f}\|_\infty \leq \|f\|_1$. $\hat{f}$ is continuous.
Fourier transform on $\mathbb{R}^n$

- Initially defined only for $f \in L^1(\mathbb{R}^n)$. \( \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} \, dx \).
  
  Follows: \( \| \hat{f} \|_\infty \leq \| f \|_1 \). \( \hat{f} \) is continuous.

- Trig. polynomials are not dense anymore in the usual spaces.
Initially defined only for $f \in L^1(\mathbb{R}^n)$. 
\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} \, dx.
\]
Follows: $\|\hat{f}\|_\infty \leq \|f\|_1$. $\hat{f}$ is continuous.

Trig. polynomials are not dense anymore in the usual spaces.

But \textbf{RIEMANN-LEBESGUE} is true. First for indicator function of an interval

\[
[a_1, b_1] \times \cdots \times [a_n, b_n].
\]

Then approximate an $L^1$ function by finite linear combinations of such.
Fourier transform on $\mathbb{R}^n$

- Initially defined only for $f \in L^1(\mathbb{R}^n)$. \( \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} \, dx. \)
  
  Follows: \[ \|\hat{f}\|_{\infty} \leq \|f\|_1. \] \( \hat{f} \) is continuous.

- Trig. polynomials are not dense anymore in the usual spaces.

- But **Riemann-Lebesgue** is true. First for indicator function of an interval

  \[ [a_1, b_1] \times \cdots \times [a_n, b_n]. \]

  Then approximate an $L^1$ function by finite linear combinations of such.

- **Multi-index notation** $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$:

  - (a) \( |\alpha| = \alpha_1 + \cdots + \alpha_n. \)
  - (b) \( x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \)
  - (c) \( \partial^\alpha = (\partial/\partial_1)^{\alpha_1} \cdots (\partial/\partial_n)^{\alpha_n} \)
Initially defined only for \( f \in L^1(\mathbb{R}^n) \). \( \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} \, dx \).

Follows: \( \|\hat{f}\|_{\infty} \leq \|f\|_1 \). \( \hat{f} \) is continuous.

Trig. polynomials are not dense anymore in the usual spaces.

But Riemann-Lebesgue is true. First for indicator function of an interval

\[ [a_1, b_1] \times \cdots \times [a_n, b_n]. \]

Then approximate an \( L^1 \) function by finite linear combinations of such.

Multi-index notation \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \):

(a) \( |\alpha| = \alpha_1 + \cdots + \alpha_n \).

(b) \( x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \).

(c) \( \partial^\alpha = (\partial/\partial_1)^{\alpha_1} \cdots (\partial/\partial_n)^{\alpha_n} \).

Diff operators \( D^j \phi := \frac{1}{2\pi i} (\partial/\partial x_j) \), \( D^\alpha \phi = (1/2\pi i)^{|\alpha|} \partial^\alpha \).
Schwartz functions on \( \mathbb{R}^n \)

- \( L^p(\mathbb{R}^n) \) spaces are not nested.
Schwartz functions on $\mathbb{R}^n$

- $L^p(\mathbb{R}^n)$ spaces are not nested.
- Not clear how to define $\hat{f}$ for $f \in L^2$. 

Mihalis Kolountzakis (U. of Crete)

FT and applications

January 2006 25 / 36
Schwartz functions on $\mathbb{R}^n$

- $L^p(\mathbb{R}^n)$ spaces are not nested.
- Not clear how to define $\hat{f}$ for $f \in L^2$.
- Schwartz class $S$: those $\phi \in C^\infty(\mathbb{R}^n)$ s.t. for all multiindices $\alpha, \gamma$
  \[ \|\phi\|_{\alpha, \gamma} := \sup_{x \in \mathbb{R}^n} |x^\gamma \partial^\alpha \phi(x)| < \infty. \]
Schwartz functions on $\mathbb{R}^n$

- $L^p(\mathbb{R}^n)$ spaces are not nested.
- Not clear how to define $\hat{f}$ for $f \in L^2$.
- **Schwartz class** $\mathcal{S}$: those $\phi \in C^\infty(\mathbb{R}^n)$ s.t. for all multiindices $\alpha, \gamma$

$$\|\phi\|_{\alpha, \gamma} := \sup_{x \in \mathbb{R}^n} |x^\gamma \partial^\alpha \phi(x)| < \infty.$$ 

- The $\|\phi\|_{\alpha, \gamma}$ are *seminorms*. They determine the topology of $\mathcal{S}$.
Schwartz functions on $\mathbb{R}^n$

- $L^p(\mathbb{R}^n)$ spaces are not nested.
- Not clear how to define $\hat{f}$ for $f \in L^2$.
- **Schwartz class $S$:** those $\phi \in C^\infty(\mathbb{R}^n)$ s.t. for all multiindices $\alpha, \gamma$

$$\|\phi\|_{\alpha, \gamma} := \sup_{x \in \mathbb{R}^n} |x^\gamma \partial^\alpha \phi(x)| < \infty.$$  

- The $\|\phi\|_{\alpha, \gamma}$ are seminorms. They determine the topology of $S$.

- $C_0^\infty(\mathbb{R}^n) \subseteq S$
Schwartz functions on $\mathbb{R}^n$:

- $L^p(\mathbb{R}^n)$ spaces are not nested.
- Not clear how to define $\hat{f}$ for $f \in L^2$.
- **Schwartz class** $S$: those $\phi \in C^\infty(\mathbb{R}^n)$ s.t. for all multiindices $\alpha, \gamma$
  \[
  \|\phi\|_{\alpha, \gamma} := \sup_{x \in \mathbb{R}^n} |x^{\gamma} \partial^{\alpha} \phi(x)| < \infty.
  \]
  The $\|\phi\|_{\alpha, \gamma}$ are *seminorms*. They determine the topology of $S$.
- $C^\infty_0(\mathbb{R}^n) \subseteq S$
- Easy to see that $\hat{D^j(\phi)}(\xi) = \xi_j \hat{\phi}(\xi)$ and $\hat{x^j \phi}(\xi) = -\hat{D^j(\phi)}(\xi)$.
  More generally $\hat{\xi^\alpha D^\gamma \phi} = \hat{D^\alpha(\xi) \gamma \phi} = \hat{D^\alpha(-x) \gamma \phi}$.
Schwartz functions on $\mathbb{R}^n$

- $L^p(\mathbb{R}^n)$ spaces are not nested.
- Not clear how to define $\hat{f}$ for $f \in L^2$.
- Schwartz class $S$: those $\phi \in C^\infty(\mathbb{R}^n)$ s.t. for all multiindices $\alpha, \gamma$
  \[
  \|\phi\|_{\alpha, \gamma} := \sup_{x \in \mathbb{R}^n} |x^\gamma \partial^\alpha \phi(x)| < \infty.
  \]
- The $\|\phi\|_{\alpha, \gamma}$ are seminorms. They determine the topology of $S$.
- $C_0^\infty(\mathbb{R}^n) \subseteq S$
- Easy to see that $\widehat{D^j(\phi)}(\xi) = \xi_j \hat{\phi}(\xi)$ and $\widehat{x^j \phi(x)}(\xi) = -D^j \hat{\phi}(\xi)$.
  More generally $\xi^\alpha D^\gamma \hat{\phi}(\xi) = D^\alpha (-x)^\gamma \hat{\phi}(x)(\xi)$.
- $\phi \in S \implies \hat{\phi} \in S$ (smoothness $\implies$ decay, decay $\implies$ smoothness)
Schwartz functions on $\mathbb{R}^n$

- $L^p(\mathbb{R}^n)$ spaces are not nested.
- Not clear how to define $\hat{f}$ for $f \in L^2$.
- **Schwartz class $S$:** those $\phi \in C^\infty(\mathbb{R}^n)$ s.t. for all multiindices $\alpha, \gamma$
  \[
  \|\phi\|_{\alpha, \gamma} := \sup_{x \in \mathbb{R}^n} |x^\gamma \partial^\alpha \phi(x)| < \infty.
  \]

  The $\|\phi\|_{\alpha, \gamma}$ are **seminorms**. They determine the topology of $S$.
- $C_0^\infty(\mathbb{R}^n) \subseteq S$
- Easy to see that $D_j(\hat{\phi})(\xi) = \xi_j \hat{\phi}(\xi)$ and $x_j \hat{\phi}(x)(\xi) = -D_j \hat{\phi}(\xi)$.
  More generally $\xi^\alpha D^\gamma \hat{\phi}(\xi) = D^\alpha(-x)^\gamma \phi(x)(\xi)$.
- $\phi \in S \implies \hat{\phi} \in S$ (smoothness $\implies$ decay, decay $\implies$ smoothness)
- **Fourier inversion formula:** $\phi(x) = \int \hat{\phi}(\xi) e^{2\pi i \xi \cdot x} \, d\xi$.
  Can also write as $\hat{\phi}(-x) = \phi(x)$.
Schwartz functions on $\mathbb{R}^n$

- $L^p(\mathbb{R}^n)$ spaces are not nested.
- Not clear how to define $\hat{f}$ for $f \in L^2$.
- **Schwartz class $S$:** those $\phi \in C^\infty(\mathbb{R}^n)$ s.t. for all multiindices $\alpha, \gamma$

$$\|\phi\|_{\alpha, \gamma} := \sup_{x \in \mathbb{R}^n} |x^\gamma \partial^\alpha \phi(x)| < \infty.$$ 

The $\|\phi\|_{\alpha, \gamma}$ are *seminorms*. They determine the topology of $S$.

- $C^\infty_0(\mathbb{R}^n) \subseteq S$
- Easy to see that $D^j(\hat{\phi})(\xi) = \xi_j \hat{\phi}(\xi)$ and $x_j \hat{\phi}(x)(\xi) = -D^j \hat{\phi}(\xi)$.
- More generally $\xi^\alpha D^\gamma \hat{\phi}(\xi) = D^\alpha (-x)^\gamma \phi(x)(\xi)$.

- $\phi \in S \implies \hat{\phi} \in S$ (smoothness $\implies$ decay, decay $\implies$ smoothness)

**Fourier inversion formula:** $\phi(x) = \int \hat{\phi}(\xi) e^{2\pi i \xi x} \, d\xi$.

Can also write as $\hat{\phi}(-x) = \phi(x)$.

- We first show its validity for $\phi \in S$. 

Fourier inversion formula on $S$

- $\hat{\text{shifting}}$: $f, g \in L^1(\mathbb{R}^n) \implies \int f \hat{g} = \int \hat{f} g$ (Fubini)

Define the Gaussian function $g(x) = (2\pi)^{-n/2} e^{-|x|^2/2}$, $x \in \mathbb{R}^n$.

This normalization gives $\int g(x) = \int |x|^2 g(x) = 1$.

Using Cauchy's integral formula for analytic functions we prove $\hat{g}(\xi) = (2\pi)^{-n/2} g(2\pi\xi)$. The Fourier inversion formula holds.

Write $g(\cdot) = \epsilon^{-n} g(x/\epsilon)$, an approximate identity.

We have $\hat{g}(\epsilon\cdot) = (2\pi)^{-n/2} g(2\pi\epsilon\cdot)$, $\lim_{\epsilon \to 0} \hat{g}(\epsilon\cdot) = 1$.

$\hat{\phi}(\cdot - x) = \int e^{2\pi i \xi \cdot x} \hat{\phi}(\xi) d\xi = \lim_{\epsilon \to 0} \int e^{2\pi i \xi \cdot x} \hat{\phi}(\xi) \hat{g}(\epsilon\cdot) d\xi$ (dom. conv.)

$= \lim_{\epsilon \to 0} \int \hat{\phi}(\cdot + y) \hat{g}(\epsilon \cdot) dy$ (FT of translation)

$= \lim_{\epsilon \to 0} \int \phi(x+y) g(\epsilon y) dy$ (FT inversion for $g(\epsilon \cdot)$).

$= \phi(x)$ (by FT inversion, $g(\epsilon \cdot)$ is an approximate identity).
Fourier inversion formula on $S$

- Shifting: $f, g \in L^1(\mathbb{R}^n) \implies \int f \hat{g} = \int \hat{f} g$ (Fubini)
- Define the Gaussian function $g(x) = (2\pi)^{-n/2}e^{-|x|^2/2}$, $x \in \mathbb{R}^n$. This normalization gives $\int g(x) = \int |x|^2 g(x) = 1$. 

Mihalis Kolountzakis (U. of Crete)
Fourier inversion formula on $S$

- ^ shifting: $f, g \in L^1(\mathbb{R}^n) \implies \int f \hat{g} = \int \hat{f} g$ (Fubini)
- Define the **Gaussian** function $g(x) = (2\pi)^{-n/2} e^{-|x|^2/2}$, $x \in \mathbb{R}^n$. This normalization gives $\int g(x) = \int |x|^2 g(x) = 1$.
- Using **Cauchy**’s integral formula for analytic functions we prove $\hat{g}(\xi) = (2\pi)^{n/2} g(2\pi \xi)$. The Fourier inversion formula holds.
Fourier inversion formula on $S$

- Shifting: $f, g \in L^1(\mathbb{R}^n) \implies \int f \hat{g} = \int \hat{f} g$ (Fubini)
- Define the Gaussian function $g(x) = (2\pi)^{-n/2} e^{-|x|^2/2}$, $x \in \mathbb{R}^n$. This normalization gives $\int g(x) = \int |x|^2 g(x) = 1$.
- Using Cauchy's integral formula for analytic functions we prove $\hat{g}(\xi) = (2\pi)^{n/2} g(2\pi \xi)$. The Fourier inversion formula holds.
- Write $g_\epsilon(x) = \epsilon^{-n} g(x/\epsilon)$, an approximate identity.
Fourier inversion formula on $S$

- $\hat{\text{shifting}}$: $f, g \in L^1(\mathbb{R}^n) \implies \int f \hat{g} = \int \hat{f} g$ (Fubini)

- Define the **Gaussian** function $g(x) = (2\pi)^{-n/2} e^{-|x|^2/2}$, $x \in \mathbb{R}^n$. This normalization gives $\int g(x) = \int |x|^2 g(x) = 1$.

- Using **Cauchy**’s integral formula for analytic functions we prove $\hat{g}(\xi) = (2\pi)^{n/2} g(2\pi \xi)$. The Fourier inversion formula holds.

- Write $g_\epsilon(x) = \epsilon^{-n} g(x/\epsilon)$, an approximate identity.

- We have $\hat{g}_\epsilon(\xi) = (2\pi)^{n/2} g(2\pi \epsilon \xi)$, $\lim_{\epsilon \to 0} \hat{g}_\epsilon(\xi) = 1$. 
Fourier inversion formula on $S$

- shifting: $f, g \in L^1(\mathbb{R}^n) \implies \int f \hat{g} = \int \hat{f} g$ (Fubini)

- Define the **Gaussian** function $g(x) = (2\pi)^{-n/2} e^{-|x|^2/2}$, $x \in \mathbb{R}^n$. This normalization gives $\int g(x) = \int |x|^2 g(x) = 1$.

- Using **Cauchy**'s integral formula for analytic functions we prove $\hat{g}(\xi) = (2\pi)^n/2 g(2\pi \xi)$. The Fourier inversion formula holds.

- Write $g_\epsilon(x) = \epsilon^{-n} g(x/\epsilon)$, an approximate identity.

- We have $\hat{g}_\epsilon(\xi) = (2\pi)^n/2 g(2\pi \epsilon \xi)$, $\lim_{\epsilon \to 0} \hat{g}_\epsilon(\xi) = 1$.

- $\hat{\phi}(-x) = \int e^{2\pi i \xi \cdot x} \hat{\phi}(\xi) \, d\xi = \lim_{\epsilon \to 0} \int e^{2\pi i \xi \cdot x} \hat{\phi}(\xi) \hat{g}_\epsilon(\xi) \, d\xi$ (dom. conv.)
Fourier inversion formula on $S$

- Shifting: $f, g \in L^1(\mathbb{R}^n) \implies \int f \hat{g} = \int \hat{f} g$ (Fubini)

- Define the Gaussian function $g(x) = (2\pi)^{-n/2} e^{-|x|^2/2}, x \in \mathbb{R}^n$. This normalization gives $\int g(x) = \int |x|^2 g(x) = 1$.

- Using Cauchy's integral formula for analytic functions we prove $\hat{g}(\xi) = (2\pi)^{n/2} g(2\pi \xi)$. The Fourier inversion formula holds.

- Write $g_\epsilon(x) = \epsilon^{-n} g(x/\epsilon)$, an approximate identity.

- We have $\hat{g}_\epsilon(\xi) = (2\pi)^{n/2} g(2\pi \epsilon \xi), \lim_{\epsilon \to 0} \hat{g}_\epsilon(\xi) = 1$.

- $\hat{\phi}(-x) = \int e^{2\pi i \xi x} \hat{\phi}(\xi) d\xi = \lim_{\epsilon \to 0} \int e^{2\pi i \xi x} \hat{\phi}(\xi) \hat{g}_\epsilon(\xi) d\xi$ (dom. conv.)

- $= \lim_{\epsilon \to 0} \int \phi(\cdot + x)(\xi) \hat{g}_\epsilon(\xi) d\xi$ (FT of translation)
Fourier inversion formula on $S$

- shifting: \( f, g \in L^1(\mathbb{R}^n) \implies \int f \hat{g} = \int \hat{f} g \) (Fubini)

- Define the Gaussian function \( g(x) = (2\pi)^{-n/2} e^{-|x|^2/2}, x \in \mathbb{R}^n \). This normalization gives \( \int g(x) = \int |x|^2 g(x) = 1 \).

- Using Cauchy's integral formula for analytic functions we prove \( \hat{g}(\xi) = (2\pi)^{n/2} g(2\pi \xi) \). The Fourier inversion formula holds.

- Write \( g_\epsilon(x) = \epsilon^{-n} g(x/\epsilon) \), an approximate identity.

- We have \( \hat{g}_\epsilon(\xi) = (2\pi)^{n/2} g(2\pi \epsilon \xi) \), \( \lim_{\epsilon \to 0} \hat{g}_\epsilon(\xi) = 1 \).

- \( \hat{\phi}(-x) = \int e^{2\pi i \xi x} \hat{\phi}(\xi) \, d\xi = \lim_{\epsilon \to 0} \int e^{2\pi i \xi x} \hat{\phi}(\xi) \hat{g}_\epsilon(\xi) \, d\xi \) (dom. conv.)

- \( = \lim_{\epsilon \to 0} \int \hat{\phi}(\cdot + x)(\xi) \hat{g}_\epsilon(\xi) \, d\xi \) (FT of translation)

- \( = \lim_{\epsilon \to 0} \int \phi(x + y) \hat{g}_\epsilon(y) \, dy \) (\(^\wedge\) shifting)
Fourier inversion formula on $S$

- $\widehat{\text{shifting}}$: $f, g \in L^1(\mathbb{R}^n) \implies \int f \hat{g} = \int \hat{f} g$ (Fubini)

- Define the **Gaussian** function $g(x) = (2\pi)^{-n/2} e^{-|x|^2/2}$, $x \in \mathbb{R}^n$. This normalization gives $\int g(x) = \int |x|^2 g(x) = 1$.

- Using **Cauchy**’s integral formula for analytic functions we prove $\hat{g}(\xi) = (2\pi)^{n/2} g(2\pi \xi)$. The Fourier inversion formula holds.

- Write $g_\epsilon(x) = \epsilon^{-n} g(x/\epsilon)$, an approximate identity.

- We have $\hat{g}_\epsilon(\xi) = (2\pi)^{n/2} g(2\pi \epsilon \xi)$, $\lim_{\epsilon \to 0} \hat{g}_\epsilon(\xi) = 1$.

- $\hat{\phi}(-x) = \int e^{2\pi i \xi x} \hat{\phi}(\xi) \, d\xi = \lim_{\epsilon \to 0} \int e^{2\pi i \xi x} \hat{\phi}(\xi) \hat{g}_\epsilon(\xi) \, d\xi$ (dom. conv.)

- $= \lim_{\epsilon \to 0} \int \hat{\phi}(\cdot + x)(\xi) \hat{g}_\epsilon(\xi) \, d\xi$ (FT of translation)

- $= \lim_{\epsilon \to 0} \int \phi(x + y) \hat{g}_\epsilon(y) \, dy$ ($\widehat{\text{shifting}}$)

- $= \lim_{\epsilon \to 0} \int \phi(x + y) g_\epsilon(-y) \, dy$ (FT inversion for $g_\epsilon$).
Fourier inversion formula on $S$

- $\hat{\text{shifting}}$: $f, g \in L^1(\mathbb{R}^n) \implies \int f \hat{g} = \int \hat{f} g$ (Fubini)

- Define the **Gaussian** function $g(x) = (2\pi)^{-n/2} e^{-|x|^2/2}, x \in \mathbb{R}^n$. This normalization gives $\int g(x) = \int |x|^2 g(x) = 1$.

- Using **Cauchy**’s integral formula for analytic functions we prove $\hat{g}(\xi) = (2\pi)^{n/2} g(2\pi \xi)$. The Fourier inversion formula holds.

- Write $g_\epsilon(x) = \epsilon^{-n} g(x/\epsilon)$, an approximate identity.

- We have $\hat{g}_\epsilon(\xi) = (2\pi)^{-n/2} g(2\pi \epsilon \xi), \lim_{\epsilon \to 0} \hat{g}_\epsilon(\xi) = 1$.

- $\hat{\phi}(-x) = \int e^{2\pi i \xi x} \hat{\phi}(\xi) d\xi = \lim_{\epsilon \to 0} \int e^{2\pi i \xi x} \hat{\phi}(\xi) \hat{g}_\epsilon(\xi) d\xi$ (dom. conv.)

- $= \lim_{\epsilon \to 0} \int \hat{\phi}(\cdot + x)(\xi) \hat{g}_\epsilon(\xi) d\xi$ (FT of translation)

- $= \lim_{\epsilon \to 0} \int \phi(x + y) \hat{g}_\epsilon(y) dy$ ($\hat{\text{shifting}}$)

- $= \lim_{\epsilon \to 0} \int \phi(x + y) g_\epsilon(-y) dy$ (FT inversion for $g_\epsilon$).

- $= \phi(x)$ ($g_\epsilon$ is an approximate identity)
Preservation of inner product: $\int \phi \psi = \int \hat{\phi} \hat{\psi}$, for $\phi, \psi \in S$

Fourier inversion implies $\hat{\hat{\psi}} = \overline{\psi}$. Use $\hat{\phi}$ shifting.
Preservation of inner product: $\int \phi \psi = \int \hat{\phi} \hat{\psi}$, for $\phi, \psi \in S$.

Fourier inversion implies $\hat{\psi} = \overline{\psi}$. Use $\hat{\cdot}$ shifting.

**Parseval**: $f \in S$: $\|f\|_2 = \|\hat{f}\|_2$. 

**FT on $L^2(\mathbb{R}^n)$**
Preservation of inner product: $\int \phi \overline{\psi} = \int \widehat{\phi} \overline{\widehat{\psi}}$, for $\phi, \psi \in S$

Fourier inversion implies $\widehat{\psi} = \overline{\widehat{\psi}}$. Use $\widehat{\cdot}$ shifting.

**Parseval**: $f \in S$: $\|f\|_2 = \|\widehat{f}\|_2$.

$S$ dense in $L^2(\mathbb{R}^n)$: FT extends to $L^2$ and $f \rightarrow \widehat{f}$ is an isometry on $L^2$. 

By interpolation FT is defined on $L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$, and satisfies the Hausdorff-Young inequality:

$\|\widehat{f}\|_q \leq C_p \|f\|_p$, $\frac{1}{p} + \frac{1}{q} = 1$. 

Mihalis Kolountzakis (U. of Crete)
Preservation of inner product: $\int \phi \overline{\psi} = \int \hat{\phi} \overline{\hat{\psi}}$, for $\phi, \psi \in S$.

Fourier inversion implies $\hat{\psi} = \overline{\psi}$. Use $\hat{\cdot}$ shifting.

**Parseval:** $f \in S$: $\|f\|_2 = \|\hat{f}\|_2$.

$S$ dense in $L^2(\mathbb{R}^n)$: FT extends to $L^2$ and $f \rightarrow \hat{f}$ is an isometry on $L^2$.

By interpolation FT is defined on $L^p$, $1 \leq p \leq 2$, and satisfies the **Hausdorff-Young** inequality:

$$\|\hat{f}\|_q \leq C_p \|f\|_p, \quad \frac{1}{p} + \frac{1}{q} = 1.$$
Tempered distributions: \( S' \) is the space of continuous linear functionals on \( S \).
Tempered distributions

- Tempered distributions: \( S' \) is the space of continuous linear functionals on \( S \).

- FT defined on \( S' \): for \( u \in S' \) we define \( \hat{u}(\phi) = u(\hat{\phi}) \), for \( \phi \in S \).
Tempered distributions:

- $S'$ is the space of continuous linear functionals on $S$.
- FT defined on $S'$: for $u \in S'$ we define $\hat{u}(\phi) = u(\hat{\phi})$, for $\phi \in S$.
- Fourier inversion for $S'$: $u(\phi(x)) = \hat{\hat{u}}(\phi(-x))$.

Tempered measures:

$$\int (1 + |x|)^{-k} d\mu(x) < \infty,$$
for some $k \in \mathbb{N}$.

These are in $S'$.

Differentiation defined as:

$$\left(\partial^\alpha u\right)(\phi) = (-1)^{|\alpha|} u\left(\partial^\alpha \phi\right).$$

FT of $L^p$ functions or tempered measures defined in $S'$.
Tempered distributions: $S'$ is the space of continuous linear functionals on $S$.

FT defined on $S'$: for $u \in S'$ we define $\hat{u}(\phi) = u(\hat{\phi})$, for $\phi \in S$.

Fourier inversion for $S'$: $u(\phi(x)) = \hat{\hat{u}}(\phi(-x))$.

$u \rightarrow \hat{u}$ is an isomorphism on $S'$.
Tempered distributions: $S'$ is the space of continuous linear functionals on $S$.

FT defined on $S'$: for $u \in S'$ we define $\hat{u}(\phi) = u(\hat{\phi})$, for $\phi \in S$.

Fourier inversion for $S'$: $u(\phi(x)) = \hat{\hat{u}}(\phi(-x))$.

$u \rightarrow \hat{u}$ is an isomorphism on $S'$.

$1 \leq p \leq \infty : L^p \subseteq S'$. If $f \in L^p$ this mapping is in $S'$:

$$\phi \rightarrow \int f \phi$$

Also $S \subseteq S'$.
Tempered distributions: $S'$ is the space of continuous linear functionals on $S$.

FT defined on $S'$: for $u \in S'$ we define $\hat{u}(\phi) = u(\hat{\phi})$, for $\phi \in S$.

Fourier inversion for $S'$: $u(\phi(x)) = \hat{\hat{u}}(\phi(-x))$.

$u \rightarrow \hat{u}$ is an isomorphism on $S'$.

$1 \leq p \leq \infty : L^p \subseteq S'$. If $f \in L^p$ this mapping is in $S'$:

$$\phi \rightarrow \int f \phi$$

Also $S \subseteq S'$.

Tempered measures: $\int (1 + |x|)^{-k} \, d|\mu|(x) < \infty$, for some $k \in \mathbb{N}$.

These are in $S'$. 

Tempered distributions: $S'$ is the space of continuous linear functionals on $S$.

FT defined on $S'$: for $u \in S'$ we define $\hat{u}(\phi) = u(\hat{\phi})$, for $\phi \in S$.

Fourier inversion for $S'$: $u(\phi(x)) = \hat{\hat{u}}(\phi(-x))$.

$u \rightarrow \hat{u}$ is an isomorphism on $S'$

$1 \leq p \leq \infty : L^p \subseteq S'$. If $f \in L^p$ this mapping is in $S'$:

$$\phi \rightarrow \int f \phi$$

Also $S \subseteq S'$.

Tempered measures: $\int (1 + |x|)^{-k} \, d|\mu|(x) < \infty$, for some $k \in \mathbb{N}$.

These are in $S'$.

Differentiation defined as: $(\partial^\alpha u)(\phi) = (-1)^{|\alpha|} u(\partial^\alpha \phi)$. 
Tempered distributions: \( S' \) is the space of continuous linear functionals on \( S \).

FT defined on \( S' \): for \( u \in S' \) we define \( \hat{u}(\phi) = u(\hat{\phi}) \), for \( \phi \in S \).

Fourier inversion for \( S' \): \( u(\phi(x)) = \hat{\hat{u}}(\phi(-x)) \).

\( u \to \hat{u} \) is an isomorphism on \( S' \)

1 ≤ \( p \) ≤ \( \infty \) : \( L^p \subseteq S' \). If \( f \in L^p \) this mapping is in \( S' \):

\[
\phi \to \int f \phi
\]

Also \( S \subseteq S' \).

Tempered measures: \( \int (1 + |x|)^{-k} \, d|\mu|(x) < \infty \), for some \( k \in \mathbb{N} \). These are in \( S' \).

Differentiation defined as: \( (\partial^\alpha u)(\phi) = (-1)^{|\alpha|} u(\partial^\alpha \phi) \).

FT of \( L^p \) functions or tempered measures defined in \( S' \)
Examples of tempered distributions and their FT

- \( u = \delta_0, \hat{u} = 1 \)
Examples of tempered distributions and their FT

- $u = \delta_0$, $\hat{u} = 1$
- $u = D^\alpha \delta_0$. To find its FT

$$
\hat{D^\alpha \delta_0}(\phi) = (D^\alpha \delta_0)(\hat{\phi}) = (-1)^{|\alpha|} \delta_0(D^\alpha \hat{\phi}) = (-1)^{|\alpha|} \delta_0((-x)^\alpha \phi(x))
$$

$$
= (-1)^{|\alpha|} \delta_0((-x)^\alpha \phi(x)) = \int x^\alpha \phi(x)
$$
Examples of tempered distributions and their FT

- $u = \delta_0$, $\hat{u} = 1$
- $u = D^\alpha \delta_0$. To find its FT
  \[
  \hat{D^\alpha \delta_0}(\phi) = (D^\alpha \delta_0)(\hat{\phi}) = (-1)^{|\alpha|} \delta_0(D^\alpha \hat{\phi}) = (-1)^{|\alpha|} \delta_0((-x)^\alpha \phi(x))
  \]
  \[
  = (-1)^{|\alpha|} \delta_0((-x)^\alpha \phi(x)) = \int x^\alpha \phi(x)
  \]
- So $\hat{D^\alpha \delta_0} = x^\alpha$. 
Examples of tempered distributions and their FT

- $u = \delta_0$, $\hat{u} = 1$
- $u = D^\alpha \delta_0$. To find its FT

$$\hat{D^\alpha \delta_0}(\phi) = (D^\alpha \delta_0)(\hat{\phi}) = (-1)^{|\alpha|} \delta_0(D^\alpha \hat{\phi}) = (-1)^{|\alpha|} \delta_0((-x)^\alpha \phi(x))$$

$$= (-1)^{|\alpha|} \delta_0((-x)^\alpha \phi(x)) = \int x^\alpha \phi(x)$$

- So $\hat{D^\alpha \delta_0} = x^\alpha$.
- $u = x^\alpha$, $\hat{u} = D^\alpha \delta_0$
Examples of tempered distributions and their FT

- $u = \delta_0$, $\hat{u} = 1$
- $u = D^\alpha \delta_0$. To find its FT

$$\widehat{D^\alpha \delta_0}(\phi) = (D^\alpha \delta_0)(\hat{\phi}) = (-1)^{|\alpha|} \delta_0(D^\alpha \hat{\phi}) = (-1)^{|\alpha|} \delta_0((-x)^\alpha \phi(x))$$

$$= (-1)^{|\alpha|} \delta_0((-x)^\alpha \phi(x)) = \int x^\alpha \phi(x)$$

- So $\widehat{D^\alpha \delta_0} = x^\alpha$.
- $u = x^\alpha$, $\hat{u} = D^\alpha \delta_0$
- $u = \sum_{j=1}^{J} a_j \delta_{p_j}$, $\hat{u}(\xi) = \sum_{j=1}^{J} a_j e^{2\pi ip_j \xi}$. 

Poisson Summation Formula (PSF):
Examples of tempered distributions and their FT

- \( u = \delta_0, \hat{u} = 1 \)
- \( u = D^\alpha \delta_0 \). To find its FT

\[
\hat{D^\alpha \delta_0}(\phi) = (D^\alpha \delta_0)(\hat{\phi}) = (-1)^{|\alpha|} \delta_0(D^\alpha \hat{\phi}) = (-1)^{|\alpha|} \delta_0((-x)^\alpha \phi(x)) \\
= (-1)^{|\alpha|} \delta_0((-x)^\alpha \phi(x)) = \int x^\alpha \phi(x)
\]

- So \( \hat{D^\alpha \delta_0} = x^\alpha \).
- \( u = x^\alpha, \hat{u} = D^\alpha \delta_0 \)
- \( u = \sum_{j=1}^{J} a_j \delta_{p_j}, \hat{u}(\xi) = \sum_{j=1}^{J} a_j e^{2\pi ip_j \xi} \).
- **Poisson Summation Formula (PSF):** \( u = \sum_{k \in \mathbb{Z}^n} \delta_k, \hat{u} = u \)
\( \phi \in S. \) Define \( g(x) = \sum_{k \in \mathbb{Z}^n} \phi(x + k). \)
Proof of the Poisson Summation Formula

- \( \phi \in S \). Define \( g(x) = \sum_{k \in \mathbb{Z}^n} \phi(x + k) \).
- \( g \) has \( \mathbb{Z}^n \) as a period lattice: \( g(x + k) = g(x), \ x \in \mathbb{R}^n, k \in \mathbb{Z}^n \).
Proof of the Poisson Summation Formula

- \( \phi \in S \). Define \( g(x) = \sum_{k \in \mathbb{Z}^n} \phi(x + k) \).
- \( g \) has \( \mathbb{Z}^n \) as a period lattice: \( g(x + k) = g(x), \ x \in \mathbb{R}^n, k \in \mathbb{Z}^n \).
- The periodization \( g \) may be viewed as \( g : \mathbb{T}^n \rightarrow \mathbb{C} \).
Proof of the Poisson Summation Formula

- \( \phi \in S \). Define \( g(x) = \sum_{k \in \mathbb{Z}^n} \phi(x + k) \).
- \( g \) has \( \mathbb{Z}^n \) as a period lattice: \( g(x + k) = g(x), \ x \in \mathbb{R}^n, k \in \mathbb{Z}^n \).
- The periodization \( g \) may be viewed as \( g : \mathbb{T}^n \rightarrow \mathbb{C} \).
- FT of \( g \) lives on \( \mathbb{T}^n = \mathbb{Z}^n \). The Fourier coefficients are

\[
\hat{g}(k) = \int_{\mathbb{T}^n} \sum_{m \in \mathbb{Z}^n} \phi(x + m) e^{-2\pi ikx} \, dx = \hat{\phi}(k).
\]
\[ \phi \in S. \text{ Define } g(x) = \sum_{k \in \mathbb{Z}^n} \phi(x + k). \]

\[ g \text{ has } \mathbb{Z}^n \text{ as a period lattice: } g(x + k) = g(x), \ x \in \mathbb{R}^n, k \in \mathbb{Z}^n. \]

\[ \text{The periodization } g \text{ may be viewed as } g : \mathbb{T}^n \to \mathbb{C}. \]

\[ \text{FT of } g \text{ lives on } \hat{\mathbb{T}}^n = \mathbb{Z}^n. \text{ The Fourier coefficients are} \]

\[ \hat{g}(k) = \int_{\mathbb{T}^n} \sum_{m \in \mathbb{Z}^n} \phi(x + m)e^{-2\pi i k x} \, dx = \hat{\phi}(k). \]

\[ \text{From decay of } \hat{\phi} \text{ follows that the FS of } g(x) \text{ converges absolutely and uniformly and} \]

\[ g(x) = \sum_{k \in \mathbb{Z}^n} \hat{\phi}(k)e^{2\pi i k x}. \]
Proof of the Poisson Summation Formula

- $\phi \in S$. Define $g(x) = \sum_{k \in \mathbb{Z}^n} \phi(x + k)$.
- $g$ has $\mathbb{Z}^n$ as a period lattice: $g(x + k) = g(x)$, $x \in \mathbb{R}^n, k \in \mathbb{Z}^n$.
- The periodization $g$ may be viewed as $g : \mathbb{T}^n \to \mathbb{C}$.
- FT of $g$ lives on $\mathbb{Z}^n = \mathbb{T}^n$. The Fourier coefficients are

$$\hat{g}(k) = \int_{\mathbb{T}^n} \sum_{m \in \mathbb{Z}^n} \phi(x + m) e^{-2\pi i k x} \, dx = \hat{\phi}(k).$$

- From decay of $\hat{\phi}$ follows that the FS of $g(x)$ converges absolutely and uniformly and

$$g(x) = \sum_{k \in \mathbb{Z}^n} \hat{\phi}(k) e^{2\pi i k x}.$$

- $x = 0$ gives the PSF: $\sum_{k \in \mathbb{Z}^n} \phi(k) = \sum_{k \in \mathbb{Z}^n} \hat{\phi}(k)$. 
Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is a non-singular linear operator. $u \in S'$, $\nu = u \circ T$. 

Change of variables formula for integration implies $\hat{\nu} = \frac{1}{|\det T|} \hat{u} \circ T^{-\top}$.

Write $\mathbb{R}^n = H \oplus H^\perp$, $H$ a linear subspace.

Projection onto subspace defined by $(\pi_H f)(h) = \int_h \hat{f}(h+x) \, dx$, $(h \in H)$.

For $\xi \in H$: $\hat{\pi_H f}(\xi) = \hat{f}(\xi)$ (Fubini).
Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is a non-singular linear operator. $u \in S'$, $v = u \circ T$.

Change of variables formula for integration implies

$$\hat{v} = \frac{1}{|\det T|} \hat{u} \circ T^{-\top}.$$
Suppose $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a non-singular linear operator. $u \in S'$, $v = u \circ T$.

Change of variables formula for integration implies

$$\hat{v} = \frac{1}{|\det T|} \hat{u} \circ T^{-\top}.$$ 

Write $\mathbb{R}^n = H \oplus H^\perp$, $H$ a linear subspace.
Suppose $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a non-singular linear operator. $u \in S'$, $v = u \circ T$.

Change of variables formula for integration implies

$$\hat{v} = \frac{1}{|\det T|} \hat{u} \circ T^{-\top}.$$ 

Write $\mathbb{R}^n = H \oplus H^\perp$, $H$ a linear subspace.

Projection onto subspace defined by

$$(\pi_H f)(h) := \int_{H^\perp} f(h + x) \, dx, \quad (h \in H).$$
Suppose $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a non-singular linear operator. $u \in S'$, $v = u \circ T$.

Change of variables formula for integration implies

$$\hat{v} = \frac{1}{|\det T|} \hat{u} \circ T^{-T}.$$ 

Write $\mathbb{R}^n = H \oplus H^\perp$, $H$ a linear subspace.

Projection onto subspace defined by

$$(\pi_H f)(h) := \int_{H^\perp} f(h + x) \, dx, \quad (h \in H).$$

For $\xi \in H$: $\hat{\pi_H f}(\xi) = \hat{f}(\xi)$ (Fubini).
Compact support: \( f : \mathbb{R} \rightarrow \mathbb{C}, f \in L^1(\mathbb{R}), f(x) = 0 \) for \(|x| > R\).
Analyticity of the FT

- Compact support: \( f : \mathbb{R} \to \mathbb{C}, f \in L^1(\mathbb{R}), f(x) = 0 \) for \( |x| > R \).
- FT defined by

\[
\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} \, dx.
\]
Analyticity of the FT

- Compact support: \( f: \mathbb{R} \rightarrow \mathbb{C}, \ f \in L^1(\mathbb{R}), \ f(x) = 0 \) for \( |x| > R \).
- FT defined by
  \[
  \hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i \xi x} \, dx.
  \]
- Allow \( \xi \in \mathbb{C}, \ \xi = s + it \) in the formula.
  \[
  \hat{f}(s + it) = \int_{\mathbb{R}} f(x)e^{2\pi tx}e^{-2\pi isx} \, dx
  \]
Analyticity of the FT

- **Compact support**: \( f : \mathbb{R} \to \mathbb{C}, \ f \in L^1(\mathbb{R}), \ f(x) = 0 \text{ for } |x| > R. \)
- FT defined by
  \[
  \hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} \, dx.
  \]
- Allow \( \xi \in \mathbb{C}, \ \xi = s + it \) in the formula.
  \[
  \hat{f}(s + it) = \int_{\mathbb{R}} f(x) e^{2\pi t x} e^{-2\pi isx} \, dx
  \]
- Compact support implies \( f(x)e^{2\pi t x} \in L^1(\mathbb{R}), \) so integral is defined.
Compact support: \( f: \mathbb{R} \to \mathbb{C}, \ f \in L^1(\mathbb{R}), \ f(x) = 0 \text{ for } |x| > R. \)

FT defined by

\[
\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} \, dx.
\]

Allow \( \xi \in \mathbb{C}, \ \xi = s + it \) in the formula.

\[
\hat{f}(s + it) = \int_{\mathbb{R}} f(x) e^{2\pi tx} e^{-2\pi i s x} \, dx
\]

Compact support implies \( f(x) e^{2\pi t x} \in L^1(\mathbb{R}), \) so integral is defined.

Since \( e^{-2\pi i x \xi} \) is analytic for all \( \xi \in \mathbb{C}, \) so is \( \hat{f}(\xi). \)
Analyticity of the FT

- **Compact support**: $f : \mathbb{R} \to \mathbb{C}$, $f \in L^1(\mathbb{R})$, $f(x) = 0$ for $|x| > R$.

- FT defined by
  \[
  \hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} \, dx.
  \]

- Allow $\xi \in \mathbb{C}$, $\xi = s + it$ in the formula.
  \[
  \hat{f}(s + it) = \int_{\mathbb{R}} f(x) e^{2\pi tx} e^{-2\pi isx} \, dx
  \]

- Compact support implies $f(x) e^{2\pi tx} \in L^1(\mathbb{R})$, so integral is defined.

- Since $e^{-2\pi ix\xi}$ is analytic for all $\xi \in \mathbb{C}$, so is $\hat{f}(\xi)$.

- **Paley–Wiener**: $f \in L^2(\mathbb{R})$. The following are equivalent:
  (a) $f$ is the restriction on $\mathbb{R}$ of a function $F$ holomorphic in the strip $\{z : |\Im z| < a\}$ which satisfies
  \[
  \int |F(x + iy)|^2 \, dx \leq C, \quad (|y| < a)
  \]
  (b) $e^{a|\xi|} \hat{f}(\xi) \in L^2(\mathbb{R})$. 
Question of \textbf{Steinhaus}: Is there $E \subseteq \mathbb{R}^2$ such that no matter how translated and rotated it always contains exactly one point with integer coordinates?
Question of Steinhaus: Is there $E \subseteq \mathbb{R}^2$ such that no matter how translated and rotated it always contains exactly one point with integer coordinates?

Two versions: $E$ is required to be measurable or not.

Non-measurable version was answered in the affirmative by Jackson and Mauldin a few years ago. Measurable version remains open.

Equivalent form ($\theta \in \mathbb{R}^\theta$ is rotation by $\theta$):

$$\sum_{k \in \mathbb{Z}^2} 1_{\mathbb{R}^\theta E}(t+k) = 1, \quad (0 \leq \theta < 2\pi, t \in \mathbb{R}^2).$$

We prove: there is no bounded measurable Steinhaus set.

Integrating (5) for $t \in [0,1]^2$ we obtain $|E| = 1$. LHS of (5) is the $\mathbb{Z}^2$-periodization of $1_{\mathbb{R}^\theta E}$. Hence $\hat{1}_{\mathbb{R}^\theta E}(k) = 0$, $k \in \mathbb{Z}^2 \setminus \{0\}$.

$\hat{1}_E(\xi) = 0$, whenever $\xi$ on a circle through a lattice point.
Question of Steinhaus: Is there $E \subseteq \mathbb{R}^2$ such that no matter how translated and rotated it always contains exactly one point with integer coordinates?

Two versions: $E$ is required to be measurable or not

Non-measurable version was answered in the affirmative by Jackson and Mauldin a few years ago.
Question of Steinhaus: Is there $E \subseteq \mathbb{R}^2$ such that no matter how translated and rotated it always contains exactly one point with integer coordinates?

Two versions: $E$ is required to be measurable or not

Non-measurable version was answered in the affirmative by Jackson and Mauldin a few years ago.

Measurable version remains open.
Application: the **Steinhaus** tiling problem

- **Question of Steinhaus:** Is there $E \subseteq \mathbb{R}^2$ such that no matter how translated and rotated it always contains exactly one point with integer coordinates?
- **Two versions:** $E$ is required to be measurable or not.
- Non-measurable version was answered in the affirmative by Jackson and Mauldin a few years ago.
- Measurable version remains open.
- Equivalent form ($R_\theta$ is rotation by $\theta$):
  \[
  \sum_{k \in \mathbb{Z}^2} 1_{R_\theta E}(t + k) = 1, \quad (0 \leq \theta < 2\pi, \ t \in \mathbb{R}^2). \quad (5)
  \]
Application: the \textbf{Steinhaus} tiling problem

- **Question of Steinhaus:** Is there $E \subseteq \mathbb{R}^2$ such that no matter how translated and rotated it always contains exactly one point with integer coordinates?
- Two versions: $E$ is required to be measurable or not
- Non-measurable version was answered in the affirmative by \textsc{Jackson} and \textsc{Mauldin} a few years ago.
- Measurable version remains open.
- Equivalent form ($R_\theta$ is rotation by $\theta$):
  \begin{equation}
  \sum_{k \in \mathbb{Z}^2} 1_{R_\theta E}(t + k) = 1, \quad (0 \leq \theta < 2\pi, \; t \in \mathbb{R}^2).
  \end{equation}
- We prove: there is no \textbf{bounded} measurable \textbf{Steinhaus} set.
Application: the Steinhaus tiling problem

- **Question of Steinhaus**: Is there $E \subseteq \mathbb{R}^2$ such that no matter how translated and rotated it always contains exactly one point with integer coordinates?

- **Two versions**: $E$ is required to be measurable or not

  - Non-measurable version was answered in the affirmative by Jackson and Mauldin a few years ago.
  - Measurable version remains open.

- **Equivalent form** ($R_{\theta}$ is rotation by $\theta$):
  \[
  \sum_{k \in \mathbb{Z}^2} 1_{R_{\theta}E}(t + k) = 1, \quad (0 \leq \theta < 2\pi, \ t \in \mathbb{R}^2).
  \] (5)

- We prove: there is no bounded measurable Steinhaus set.

- Integrating (5) for $t \in [0, 1]^2$ we obtain $|E| = 1$. 
Application: the Steinhaus tiling problem

- **Question of Steinhaus:** Is there \( E \subseteq \mathbb{R}^2 \) such that no matter how translated and rotated it always contains exactly one point with integer coordinates?
- **Two versions:** \( E \) is required to be measurable or not
- **Non-measurable version** was answered in the affirmative by Jackson and Mauldin a few years ago.
- **Measurable version** remains open.
- **Equivalent form** \((R_\theta \text{ is rotation by } \theta)\): 
  \[
  \sum_{k \in \mathbb{Z}^2} 1_{R_\theta E}(t + k) = 1, \quad (0 \leq \theta < 2\pi, \ t \in \mathbb{R}^2). \tag{5}
  \]
- We prove: there is no bounded measurable Steinhaus set.
- Integrating (5) for \( t \in [0, 1]^2 \) we obtain \(|E| = 1\).
- LHS of (5) is the \( \mathbb{Z}^2 \)-periodization of \( 1_{R_\theta E} \). Hence \( \widehat{1_{R_\theta E}}(k) = 0, \ k \in \mathbb{Z}^2 \setminus \{0\} \).
Application: the Steinhaus tiling problem

- Question of Steinhaus: Is there $E \subseteq \mathbb{R}^2$ such that no matter how translated and rotated it always contains exactly one point with integer coordinates?
- Two versions: $E$ is required to be measurable or not.
- Non-measurable version was answered in the affirmative by Jackson and Mauldin a few years ago.
- Measurable version remains open.
- Equivalent form ($R_\theta$ is rotation by $\theta$):
  \[
  \sum_{k \in \mathbb{Z}^2} 1_{R_\theta E}(t + k) = 1, \quad (0 \leq \theta < 2\pi, \ t \in \mathbb{R}^2). \tag{5}
  \]
- We prove: there is no bounded measurable Steinhaus set.
- Integrating (5) for $t \in [0, 1]^2$ we obtain $|E| = 1$.
- LHS of (5) is the $\mathbb{Z}^2$-periodization of $1_{R_\theta E}$. Hence $\hat{1}_{R_\theta E}(k) = 0$, $k \in \mathbb{Z}^2 \setminus \{0\}$.
- $\hat{1}_E(\xi) = 0$, whenever $\xi$ on a circle through a lattice point
The circles on which $\hat{1}_E$ must vanish
Consider the projection $f$ of $1_E$ on $\mathbb{R}$.
$E$ bounded $\implies f$ has compact support, say in $[-B, B]$.
Consider the projection $f$ of $1_E$ on $\mathbb{R}$. $E$ bounded $\implies$ $f$ has compact support, say in $[-B, B]$.

For $\xi \in \mathbb{R}$ we have $\hat{f}(\xi) = \hat{1}_E(\xi, 0)$, hence

$$\hat{f}\left(\sqrt{m^2 + n^2}\right) = 0, \quad (m, n) \in \mathbb{Z}^2 \setminus \{0\}.$$
Consider the projection $f$ of $1_E$ on $\mathbb{R}$. $E$ bounded $\implies f$ has compact support, say in $[-B, B]$.

For $\xi \in \mathbb{R}$ we have $\hat{f}(\xi) = \hat{1_E}(\xi, 0)$, hence

$$\hat{f}\left(\sqrt{m^2 + n^2}\right) = 0, \quad (m, n) \in \mathbb{Z}^2 \setminus \{0\}.$$ 

**Landau:** The number of integers up to $x$ which are sums of two squares is $\sim Cx/\log^{1/2} x$. 


Consider the projection $f$ of $1_E$ on $\mathbb{R}$. 

$E$ bounded $\implies f$ has compact support, say in $[-B, B]$.

For $\xi \in \mathbb{R}$ we have $\widehat{f}(\xi) = \widehat{1_E}(\xi, 0)$, hence

$$
\widehat{f}\left(\sqrt{m^2 + n^2}\right) = 0, \quad (m, n) \in \mathbb{Z}^2 \setminus \{0\}.
$$

**Landau:** The number of integers up to $x$ which are sums of two squares is $\sim Cx / \log^{1/2} x$.

Hence $\widehat{f}$ has almost $R^2$ zeros from 0 to $R$.  

\textbf{Application: the \textbf{Steinhaus} tiling problem: conclusion}
Consider the projection $f$ of $1_E$ on $\mathbb{R}$.

$E$ bounded $\implies f$ has compact support, say in $[-B, B]$.

For $\xi \in \mathbb{R}$ we have $\hat{f}(\xi) = \hat{1_E}(\xi, 0)$, hence

$$\hat{f} \left( \sqrt{m^2 + n^2} \right) = 0, \quad (m, n) \in \mathbb{Z}^2 \setminus \{0\}.$$

Landau: The number of integers up to $x$ which are sums of two squares is $\sim Cx / \log^{1/2} x$.

Hence $\hat{f}$ has almost $R^2$ zeros from 0 to $R$.

$\text{supp } f \subseteq [-B, B]$ implies $|\hat{f}(z)| \leq \|f\|_1 e^{2\pi B|z|}$, $z \in \mathbb{C}$
Consider the projection $f$ of $1_E$ on $\mathbb{R}$.

$E$ bounded $\iff f$ has compact support, say in $[-B, B]$.

For $\xi \in \mathbb{R}$ we have $\hat{f}(\xi) = \hat{1}_E(\xi, 0)$, hence

$$\hat{f} \left( \sqrt{m^2 + n^2} \right) = 0, \quad (m, n) \in \mathbb{Z}^2 \setminus \{0\}.$$ 

**Landau:** The number of integers up to $x$ which are sums of two squares is $\sim Cx/\log^{1/2} x$.

Hence $\hat{f}$ has almost $R^2$ zeros from 0 to $R$.

$\text{supp } f \subseteq [-B, B]$ implies $|\hat{f}(z)| \leq \|f\|_1 e^{2\pi B|z|}$, $z \in \mathbb{C}$

But such a function can only have $O(R)$ zeros from 0 to $R$. 


JENSEN’s formula: $F$ analytic in the disk $\{|z| \leq R\}$, $z_k$ are the zeros of $F$ in that disk. Then

$$\sum_k \log \frac{R}{|z_k|} = \int_0^1 \log |F(Re^{2\pi i \theta})| \, d\theta.$$
JENSEN’s formula: \( F \) analytic in the disk \( \{|z| \leq R\} \), \( z_k \) are the zeros of \( F \) in that disk. Then

\[
\sum_k \log \frac{R}{|z_k|} = \int_0^1 \log \left| F(Re^{2\pi i\theta}) \right| d\theta.
\]

It follows

\[
\# \{k : |z_k| \leq R/e\} \leq \int_0^1 \log \left| F(Re^{2\pi i\theta}) \right| d\theta
\]
Jensen’s formula: $F$ analytic in the disk $\{|z| \leq R\}$, $z_k$ are the zeros of $F$ in that disk. Then

$$\sum_k \log \frac{R}{|z_k|} = \int_0^1 \log |F(Re^{2\pi i \theta})| \, d\theta.$$ 

It follows

$$\#\{k : |z_k| \leq R/e\} \leq \int_0^1 \log |F(Re^{2\pi i \theta})| \, d\theta$$

Suppose $|F(z)| \leq Ae^{B|z|}$. Then RHS above is $\leq BR + \log A$. 
JENSEN’s formula: $F$ analytic in the disk $\{ |z| \leq R \}$, $z_k$ are the zeros of $F$ in that disk. Then

$$\sum_k \log \frac{R}{|z_k|} = \int_0^1 \log \left| F(Re^{2\pi i \theta}) \right| d\theta.$$ 

It follows

$$\# \{ k : |z_k| \leq R/e \} \leq \int_0^1 \log \left| F(Re^{2\pi i \theta}) \right| d\theta.$$

Suppose $|F(z)| \leq Ae^{B|z|}$. Then RHS above is $\leq BR + \log A$.

Such a function $F$ can therefore have only $O(R)$ zeros in the disk $\{ |z| \leq R \}$. 