TILING, SPECTRALITY AND APERIODICITY OF CONNECTED SETS

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Abstract. Let $\Omega \subset \mathbb{R}^d$ be a set of finite measure. The periodic tiling conjecture suggests that if Ω tiles $\mathbb{R}^d$ by translations then it admits at least one periodic tiling. Fuglede's conjecture suggests that Ω admits an orthogonal basis of exponential functions if and only if it tiles $\mathbb{R}^d$ by translations. Both conjectures are known to be false in sufficiently high dimensions, with all the so-far-known counterexamples being highly disconnected. On the other hand, both conjectures are known to be true for convex sets. In this work we study these conjectures for connected sets. We show that the periodic tiling conjecture, as well as both directions of Fuglede's conjecture are false for connected sets in sufficiently high dimensions.

Contents

2020 *Mathematics Subject Classification.* 42B10, 52C22, 52C23.
Key words and phrases. Tiling. Spectral sets. Aperiodic tiling. Einstein tiling problem.

1. Introduction

1.1. **Trading dimension for freedom in tilings by translation.** Tiling by translation is a fascinating subject with connections to several parts of analysis and number theory, as well as, of course, geometry. Restricting the motions of the tile to translations imposes a stronger structure on tilings compared with tilings where the tile (or tiles) are allowed a greater group of motions. Tilings by translation often have, or are conjectured to have, properties that more general tilings do not have. This paper focuses on two of them: periodicity and spectrality. In the first we seek to understand if a translational tile must also be able to tile in a periodic manner, a property known to fail for tilings with a larger group of motions. In the second the Fuglede conjecture identifies domains that tile with domains that admit an orthogonal basis of exponentials for their L^2 space.

It has turned out that both these properties cease to hold when the dimension is sufficiently large. It appears that the extra freedom afforded by high dimension compensates for the rigidity imposed by restricting to translations. It is exactly this phenomenon that we exploit in this paper: increasing the dimension allows us to obtain more well behaved counterexamples to the Periodic Tiling Conjecture and to the Fuglede Conjecture, namely it allows us to obtain connected sets as counterexamples.

1.2. **Tilings and periodicity.** The study of the structure of tilings goes back to Hilbert's 18th problem. This problem was later generalized to the well known "einstein[1] problem", which asks about the existence of a single shape which tiles the space but does so only in a non-periodic way. Such a tile is called "aperiodic" or an "einstein". Socolar–Taylor [ST12] constructed a planar aperiodic tile which tiles the plane by translations, rotations and also reflections, but this tile is highly disconnected. The Socolar–Taylor construction was later extended to the Schmitt–Conway–Danzer tile: A convex three-dimensional domain which tiles $\mathbb{R}^3$ aperiodically by translations, rotations and reflections [SCD93]. The einstein problem for *planar connected tiles* remained open, until very recently, when "The Hat" tile was discovered by Smith–Myers–Kaplan–Goodman-Strauss [SMKGS23a]. Moreover, in a subsequent paper, the same authors constructed a connected planar "einstein" which tiles the plane aperiodically by translations and rotations only (no reflections) [SMKGS23b]. It is known, however, that there is no *translational* einstein which is a topological disk [BN91, Ken92]. It was recently shown [GT22] that aperiodic translational tiles exist in high dimensions. The first part of this paper (Section 2) is devoted to the question whether there are any *aperiodic connected translational tiles*.

Let $\Omega \subset \mathbb{R}^d$ be a measurable set of finite, positive measure. We call Ω a translational tile of $\mathbb{R}^d$ if there exists a (countable) set $A \subset \mathbb{R}^d$ such that the family of translates of Ω along the elements of A:

$$\Omega + a,\ a \in A,$$

covers almost every point in $\mathbb{R}^d$ exactly once. The set A is then called a tiling of $\mathbb{R}^d$ by Ω, and we write:

$$\Omega \oplus A = \mathbb{R}^d.$$

[1]Here, the word "einstein" refers to "one stone" in German.

Similarly, a finite subset $F \subset \mathbb{Z}^d$ is a translational tile of $\mathbb{Z}^d$ if there exists $A \subset \mathbb{Z}^d$ such that the sets $F+a$, $a \in A$, form a partition of $\mathbb{Z}^d$, namely: $F \oplus A = \mathbb{Z}^d$. In this case, A is called a tiling of $\mathbb{Z}^d$ by F.

For $G = \mathbb{R}^d$ or $G = \mathbb{Z}^d$, a tiling A in G is said to be periodic if there exists a lattice Λ, a discrete subgroup of G containing d linearly independent elements, such that A is invariant under translations by any point in this lattice; namely

$$A + \lambda = A, \quad \lambda \in \Lambda$$

for some co-compact subgroup Λ of G. A translational tile of G is called aperiodic if none of the tilings that it admits are periodic.

In the 60's, H. Wang [W75] conjectured that any tiling, by an arbitrary finite number of tiles, in $\mathbb{Z}^2$ admits a periodic tiling. Wang also showed that if this conjecture were true, then the question whether a given collection of finite subsets of $\mathbb{Z}^2$ tiles would be algorithmically decidable: there would be an algorithm that provides an answer to this question in finite time. A few years later, Berger proved [B64, B66] a negative answer to both questions. He constructed an aperiodic tiling with 20,426 tiles: this tile-set admits tilings but none of these tilings are periodic. Then, using this construction, he also proved that tilings by multiple tiles in $\mathbb{Z}^2$ are undecidable. Since then, there has been an extensive effort to reduce the possible size of aperiodic and undecidable tile-sets, see [GT21, Table 2]. Recently, in [GT21], it was proved that tilings with two tiles are undecidable in high dimensions.

As for translational tiling by a single tile, the celebrated *periodic tiling conjecture* [GS87, LW96] asserts that there are no aperiodic translational tiles:

Conjecture 1.1 (The periodic tiling conjecture)**.** *Let $\Omega \subset \mathbb{R}^d$ be a set of finite, positive measure. If Ω tiles $\mathbb{R}^d$ by translations then it must admit at least one periodic tiling.*

The periodic tiling conjecture is known to hold in $\mathbb{R}$ [LW96], in $\mathbb{R}^2$ for topological disks [BN91, Ken92] and also for convex domains in all dimensions [M80, V54]. However, very recently the periodic tiling conjecture was disproved in high dimensions [GT22].

Since the counterexample constructed in [GT22] is disconnected, a natural followup question is whether the periodic tiling conjecture is true for connected sets[2] in all dimensions, see [GT22, Question 10.3].

Our first result gives a negative answer to this question:

Theorem 1.2. *For sufficiently large d, there exists a set Ω in $\mathbb{R}^d$ of finite measure which is the closure of its interior, such that:*

(i) Ω is connected.

(ii) Ω tiles $\mathbb{R}^d$ by translations.

(iii) If $\Omega \oplus A = \mathbb{R}^d$ then A is non-periodic.

In fact, we show that any d-dimensional disconnected counterexample to the periodic tiling conjecture Ω gives rise to a $(d+2)$-dimensional counterexample Ω', which is *connected*.

[2]To avoid trivial constructions, e.g., adding zero-measure line segments between connected components to make the set connected while trivially preserving aperiodicity, we require that the connected set is also the closure of its interior.

The proof is done by first showing that certain type of operations on a given finite set $F \subset \mathbb{Z}^d$ preserve aperiodicity, see Theorem 2.1. This latter theorem is general, and might be of independent interest. Then, we use this theorem to construct $(d+2)$-dimensional "folded briges" between the connected components of a given aperiodic tile $F \subset \mathbb{Z}^d$, while preserving its aperiodicity. Finally, we inflate the obtained $(d+2)$-dimensional aperiodic tile, to get an aperiodic connected tile in $\mathbb{R}^{d+2}$.

1.3. **Tiling and spectrality.** A measurable set $\Omega \subset \mathbb{R}^d$ of positive, finite measure is called spectral if there is a frequency set $\Lambda \subset \mathbb{R}^d$ such that the system

$$E(\Lambda) := \{e^{2\pi i \lambda \cdot x}\}_{\lambda \in \Lambda}$$

constitutes an orthogonal basis for $L^2(\Omega)$. In this case, the set Λ is called a spectrum for Ω.

The study of spectral sets goes back to Fuglede [F74], who in 1974 conjectured that spectral sets are exactly the ones which tile by translations:

Conjecture 1.3 (Fuglede's spectral sets conjecture)**.** *A set $\Omega \subset \mathbb{R}^d$ of finite, positive measure is spectral if and only if it tiles space by translations.*

Fuglede's conjecture motivated an extensive study of the nature of the connection between the two properties: The analytic property of spectrality and the geometric property of tiling by translations. Throughout the years many positive results towards the conjecture have been obtained, see [KM10, Section 4] and the references mentioned there. In particular, the conjecture is known to hold for convex domains in all dimensions [IKT03, GL17, LM19]. Nevertheless, in 2004, Tao discovered that there exist counterexamples to Fuglede's conjecture. In [T04], he constructed examples of sets $\Omega \subset \mathbb{R}^d$, for any $d \geq 5$, which are spectral, but cannot tile by translations. Subsequently, by an enrichment of Tao's approach, examples of translational tiles which are not spectral were also constructed, and eventually the dimension in these examples was reduced down to $d \geq 3$ [KM1, KM2] (see [KM10, Section 4] for more references). All these examples arise from constructions of counterexamples to the *finite Abelian group* formulation of Fuglede's conjecture. Thus, when inflated to Euclidean space $\mathbb{R}^d$, $d \geq 3$, each of the known counterexamples is a finite union of unit cubes centered at points of the integer lattice $\mathbb{Z}^d$. However, since in all the previously known examples the arrangement of the cubes is very sparse and disconnected, Fuglede's conjecture for connected sets[3] remained open. In this paper we show that there are connected counterexamples to both directions of the conjecture.

In Section 3, from a given disconnected set in $\mathbb{R}^d$ which is spectral and does not tile, we construct a connected set in $\mathbb{R}^{d+2}$ which is spectral and does not tile:

Theorem 1.4. *For $d \geq 5$, there exists a set Ω in $\mathbb{R}^d$ of finite measure which is the closure of its interior, such that:*

(i) Ω is connected.

(ii) Ω is spectral.

(iii) Ω does not tile $\mathbb{R}^d$ by translations.

[3]As before, to avoid trivial constructions, e.g., adding zero-measure paths between connected components to make the set connected while trivially preserving its tiling and spectral properties, we require that the connected set is equal to the closure of its interior.

Similarly to the construction in Section 2, this is done by constructing "folded bridges" in $\mathbb{R}^{d+2}$ between the connected components of a given spectral set in $\mathbb{R}^d$ which is not a tile. We prove in Theorem 3.1 that this type of construction preserves spectrality as well as the tiling properties of the original set.

In Section 4, we construct, from a given disconnected set $\Omega \subset \mathbb{R}^d$ which tiles and is not spectral, a connected set in $\mathbb{R}^{\tilde{d}}$, $\tilde{d} = \tilde{d}(\Omega) > d$, which tiles and is not spectral:

Theorem 1.5. *For sufficiently large d, there exists a set Ω in $\mathbb{R}^d$ of finite measure which is the closure of its interior, such that:*

(i) Ω *is connected.*

(ii) Ω *tiles* $\mathbb{R}^d$ *by translations.*

(iii) Ω *is not spectral.*

The proof is done by iteratively constructing high dimensional "spiral bridges" between the connected components of Ω, a given finite union of unit cubes which tiles and is not spectral. In Theorem 4.1 we prove that this type of construction preserves the non-specrality as well as the tiling properties of the original set.

Theorems 3.1 and 4.1 give a range of operations on a set that preserve its spectral and tiling properties. These theorems may, therefore, be of independent interest.

1.4. **Notation and preliminaries.** Throughout this paper:

- We denote the Euclidean norm by
$$\|\cdot\|\colon \mathbb{R}^d \to [0,\infty).$$
- We denote the Lebesgue measure of a set $\Omega \subset \mathbb{R}^d$ by $|\Omega|$, and for a set $F \subset \mathbb{Z}^d$, $|F|$ denotes the cardinality of F, or, equivalently, the counting measure of F.
- For a number $r \in \mathbb{R}$, $\lfloor r \rfloor \in \mathbb{Z}$ denotes the largest integer which is smaller or equal to r, and $\lceil r \rceil$ denotes the smallest integer which is greater or equal to r.
- For sets A, B in a group G, we use the notation $A + B$ for Minkowski addition:
$$\{a + b\colon a \in A, b \in B\}$$
of A and B. For $A \subset G$ and $B \subset G'$ the set $A \times B \subset G \times G'$ is the Cartesian product:
$$\{(a,b)\colon a \in A, b \in B\}$$
of A and B.
- For a function $f\colon \mathbb{R}^d \to \mathbb{C}$ we denote
$$\{f = 0\} := \{\xi \in \mathbb{R}^d\colon f(\xi) = 0\}.$$

1.4.1. Let $\Lambda \subset \mathbb{R}^d$ be a countable set and let $\Omega \subset \mathbb{R}^d$ be measurable with positive, finite measure. Observe that the system $E(\Lambda) = \{e^{2\pi i \lambda \cdot x}\}_{\lambda \in \Lambda}$ is orthogonal in $L^2(\Omega)$ if and only if

$$(\Lambda - \Lambda) \setminus \{0\} \subset \{\widehat{\mathbb{1}_\Omega} = 0\}. \tag{1}$$

The upper density of Λ is defined as the quantity

$$\limsup_{R\to\infty} \sup_{x\in\mathbb{R}^d} \frac{|\Lambda \cap (x + [-R/2, R/2]^d)|}{R^d}$$

and the lower density of Λ is defined as

$$\liminf_{R\to\infty} \sup_{x\in\mathbb{R}^d} \frac{|\Lambda \cap (x + [-R/2, R/2]^d)|}{R^d}.$$

If the upper density of Λ is equal to its lower density, we denote both quantities by $\operatorname{dens}\Lambda$ and say that Λ has density $\operatorname{dens}\Lambda$.

The following proposition is well known in the study of spectral sets. It will be used in the proofs of Theorems 3.1 and 4.1.

Proposition 1.1. *Let $\Omega \subset \mathbb{R}^d$ be a measurable set of positive, finite measure. The following are equivalent:*

(i) Ω is spectral.

(ii) There exists $\Lambda \subset \mathbb{R}^d$ of lower density at least $|\Omega|$ such that (1) is satisfied.

Moreover, if $\Lambda \subset \mathbb{R}^d$ is a spectrum for Ω then Λ satisfies (1) and $\operatorname{dens}\Lambda = |\Omega|$.

The proof of Proposition 1.1 follows by combining [K04, Section 3.1] or [GL18, Lemma 3.1] with [K16, Theorem 1].

1.5. **Acknowledgment.** R.G. was supported by the National Science Foundation grants DMS-2242871, DMS-1926686 and by the Association of Members of the Institute for Advanced Study. M.K. was supported by the Hellenic Foundation for Research and Innovation, Project HFRI-FM17-1733 and by University of Crete Grant 4725. We thank Terence Tao for helpful suggestions to improve the exposition of the paper. We are grateful to Sha Wu of Hunan University for pointing out an error in the original "folded bridge" construction which led us to a much simplified "folded bridge".

2. Aperiodic connected tiles

Theorem 2.1 (Aperiodicity preserving operation)**.** *Let F be a finite subset of $\mathbb{Z}^d$. Define the finite set*

$$X = \left\{(v_j, s_j) : j = 0, 1, \dots, n-1\right\} \subseteq \mathbb{R}^{d+k}$$

where $v_0, \dots, v_{n-1} \in \mathbb{Z}^d$ are arbitrary and $s_0, \dots, s_{n-1}$ are n distinct points in $\mathbb{Z}^k$ such that

$$S = \{s_j \colon j = 0, 1, \dots, n-1\}$$

tiles $\mathbb{Z}^k$ by translations. Let $F' = \left(F \times \{0\}^k\right) \oplus X$. Then F' is an aperiodic tile in $\mathbb{Z}^{d+k}$ if F is an aperiodic tile of $\mathbb{Z}^d$.

Remark 2.1. If H, K are subgroups of G then $[H : H\cap K] \le [G : K]$. This implies that if $\Lambda \subseteq \mathbb{Z}^m \times \mathbb{Z}^n$ is a lattice then $\Lambda \cap \mathbb{Z}^m \times \{0\}^n$ is a lattice in $\mathbb{Z}^m \times \{0\}^n$.

Proof of Theorem 2.1. Suppose that $F \subset \mathbb{Z}^d$ is an aperiodic tile. Suppose, towards a contradiction, that F' is not aperiodic. Clearly, F' tiles $\mathbb{Z}^{d+k}$. Indeed, by assumption, there is a tiling $A \subset \mathbb{Z}^d$ of $\mathbb{Z}^d$ by F, and a tiling $T \subset \mathbb{Z}^k$ of $\mathbb{Z}^k$ by S; by construction of F' we then have that

$$A' = A \times T$$

is a tiling of $\mathbb{Z}^{d+k}$ by F'. Therefore, our assumption that F' is not aperiodic implies that there exists a periodic tiling A' of $\mathbb{Z}^{d+k}$ by F' with period lattice $G' \subseteq \mathbb{Z}^{d+k}$. Define $V = \mathbb{Z}^d \times \{0\}^k$ and

$$G = G' \cap V.$$

It follows from Remark 2.1 that G is a lattice in V. Define also the subset of V

$$A := (A' + X) \cap V. \tag{2}$$

Since for every $a' \in A'$, $x \in X$ with $a' + x \in V$ and every $g \in G$ we have $a' + x + g = (a' + g) + x = a'' + x$ for some $a'' \in A'$, we conclude that $A + G = A$, so that A is periodic in V. Thus, to arrive at a contradiction, it is enough to prove that $F \times \{0\}^k \oplus A = V$ is a tiling. Observe that for every $a' \in A'$

$$(F' + a') \cap V = (F \times \{0\}^k + X + a') \cap V = F \times \{0\}^k + \Big((X + a') \cap V\Big),$$

since $F \times \{0\}^k \subseteq V$. Thus, since

$$(F' + a') \cap V, \quad a' \in A'$$

form a tiling of V, so do the translates of $F \times \{0\}^k$ by all the points $x + a' \in V$, with $x \in X, a' \in A'$, which is exactly the set of translates A defined in (2). □

Definition 2.1. Let $B \subset \mathbb{Z}^d$. A connected component of B is a subset C of B such that $C + [0,1]^d$ is a connected component of $B + [0,1]^d$ in $\mathbb{R}^d$.

If B has a single connected component, we say that B is connected.

Remark 2.2. In our definition two points $a, b \in \mathbb{Z}^d$ are connected to each other if and only if $|a_i - b_i| \leq 1$, for all $i = 1, 2, \ldots, d$. In other words each point in $\mathbb{Z}^d$ has $3^d - 1$ neighbors.

We could strengthen the notion of connectivity for subsets of $\mathbb{Z}^d$ to demand a, so-called, $2d$-connected path from any point of the set to any other (such a path is allowed to go from any point $x \in \mathbb{Z}^d$ to any of its $2d$ neighbors along the d coordinate axes). Everything in this paper would work essentially the same.

2.1. **Folded bridge construction in $\mathbb{Z}^d$.** Let $F \subset \mathbb{Z}^d$ be finite with $m + 1 > 1$ connected components $C_0, C_1, \ldots, C_m$. Pick $m + 1$ points $a_j \in C_j$ with $a_0 = 0$ for simplicity. Then there exists a path $v_0, \ldots, v_{n-1} \in \mathbb{Z}^d$, where each v_j is a neighbor of or equal to $v_{j\pm1}$, and

$$v_0 = a_0 = 0, \quad v_{n-1} = a_m$$

and each a_j, $j = 0, 1, \ldots, m$, belongs to the path

$$\gamma : \quad v_0, v_1, \ldots, v_{n-1}.$$

Thus the path v_j connects all connected components of F. See Fig. 1.

Define the sequence $S = \Big\{s_j : \ j = 0, 1, \ldots, 2n-1\Big\} \subseteq \mathbb{Z}^2$, as follows.

$$\begin{aligned} &s_0 = (0,0), s_1 = (1,0), \ldots, s_{n-1} = (n-1,0), \\ &s_n = (n-1,1), s_{n+1} = (n-2,1), \ldots, s_{2n-2} = (1,1), s_{2n-1} = (0,1). \end{aligned} \tag{3}$$

as in Figure 2.

From $F \subseteq \mathbb{Z}^d$ we construct the set $F' \subseteq \mathbb{Z}^{d+2}$ by

$$F' = F \times \{0\}^2 + X$$

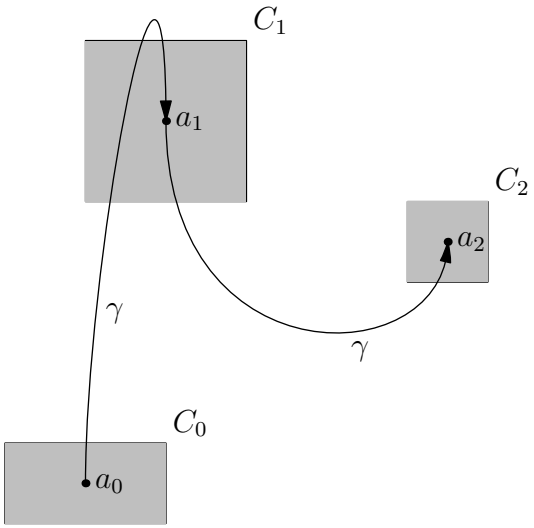

FIGURE 1. The path γ, consisting of the points $v_0, \ldots, v_{n-1}$ visits all connected components of F.

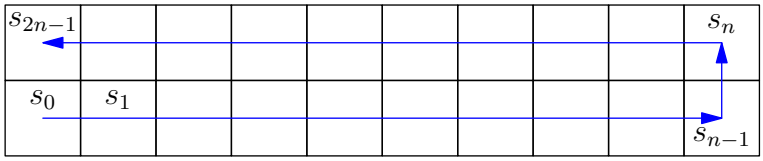

FIGURE 2. The sequence $S = \{s_0, \ldots, s_{2n-1}\} \subseteq \mathbb{Z}^2$. There are $2(m-1)$ rows in this array, each of length n.

where

$$\begin{aligned} X &= \{X_0, X_1, \ldots, X_{2n-1}\} \\ &= \{(0, s_0), (0, s_1), (0, s_2), \ldots, (0, s_{n-1}), \\ &\qquad (v_0, s_n), (v_1, s_{n+1}), \ldots, (v_{n-1}, s_{2n-1})\}. \end{aligned}$$

Notice that this is a disjoint sum since the s_j are all different (so that $|F'| = |F| \cdot 2n$).

Lemma 2.1. *The set X is connected in $\mathbb{Z}^{d+2}$.*

Proof. We first observe that for $j = 0, 1, \ldots, n-2$ the point $X_j = (0, j, 0)$ is connected to $X_{j+1} = (0, j+1, 0)$ since they only differ at one coordinate and only by 1. We also have that $X_{n-1} = (0, n-1, 0)$ is connected to $X_n = (0, n-1, 1)$ (remember $v_0 = a_0 = 0$) since they only differ at the last coordinate by 1. Finally, if $j \geq n$ then $X_j = (v_{j-n}, j, 1)$ is connected to $X_{j+1} = (v_{j-n+1}, j+1, 1)$ since their first d coordinates form two connected points in $\mathbb{Z}^d$ (since v_{j-n} is connected to v_{j-n+1}) and they also differ by 1 at the $d+1$ coordinate. □

We imagine a copy of $\mathbb{Z}^d$ "hanging" from each of the $2n$ cells in Figure 2, and, as we move from left to right and then left again, the copy of F in that copy of $\mathbb{Z}^d$ is translated by the vectors $\underbrace{0, \ldots, 0}_{n}, v_0, v_1, \ldots, v_{n-1}$.

We call this construction a "folded bridge" between the connected components $C_0, C_2, \ldots, C_m$ of F, giving F'. See Figure 3 for a visual illustration of the notion for the case $m = 2$ (three connected components).

Lemma 2.2. *F' is connected in $\mathbb{Z}^{d+2}$.*

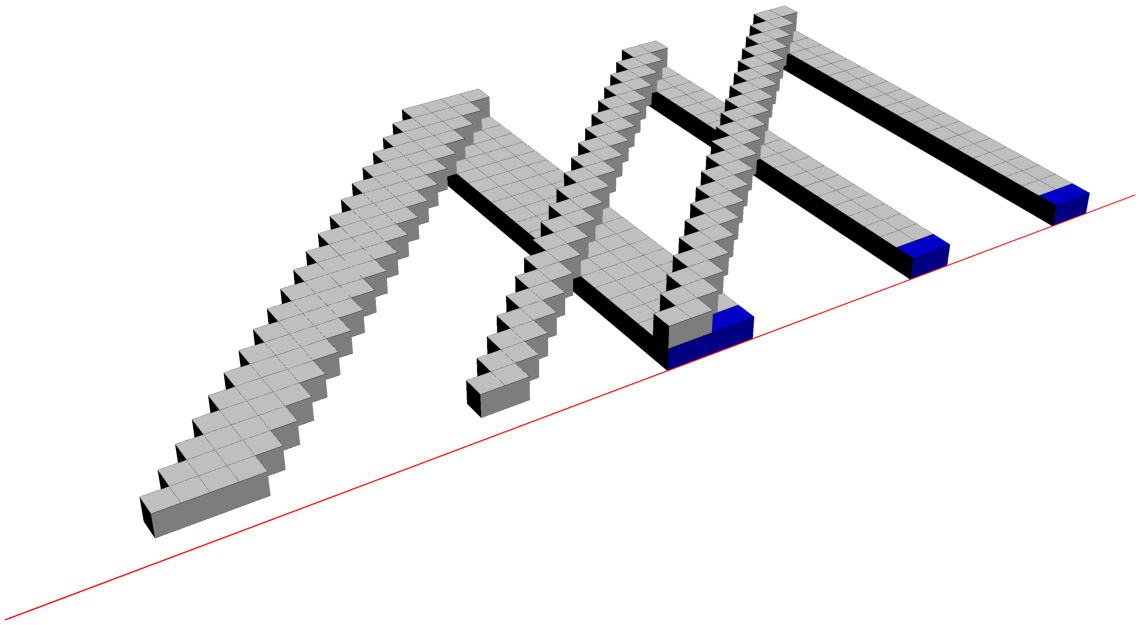

FIGURE 3. How F' is constructed from F. A folded bridge on the set F (blue is $F \times [0,1]^2$) connecting its three connected components. The red line is the ambient space for F, namely $\mathbb{Z}^d$.

Proof. We first observe that

$$F' = F \times \{0\}^2 + X = \bigcup_{j=0}^{m} (C_j \times \{0\}^2 + X),$$

and each $C_j \times \{0\}^2 + X$ is connected from Lemma 2.1 and the fact that the sum of two connected sets is connected. It remains to show that the connected sets $C_j \times \{0\}^2 + X$ connect to each other as well. We show that for $j \geq 1$ the set $C_j \times \{0\}^2 + X$ connects to $C_0 \times \{0\}^2 + X$. Indeed, there exists $k \in \{0, 1, \ldots, n-1\}$ such that $a_j = v_k$ (by the construction of the path v_j, $j = 0, \ldots, n-1$). Then (recall that $a_0 = 0$)

$$(a_j, s_{n+k}) = (a_0 + v_k, s_{n+k}) \in C_0 \times \{0\}^2 + X$$

and

$$(a_j, s_{n-k-1}) \in C_j \times \{0\}^2 + X.$$

These two points have the same first $d+1$ coordinates and differ only in the last coordinate where the first point has 1 and the second has 0. (The point s_{n+k} is right above s_{n-k-1} in Fig. 2.) □

By Theorem 2.1 we have that F' is aperiodic in $\mathbb{Z}^{d+2}$ if F is aperiodic in $\mathbb{Z}^d$.

Using this, we can finally prove Theorem 1.2:

Proof of Theorem 1.2. By [GT22, Corollary 1.5], if d is sufficiently large, we can choose a finite $F \subset \mathbb{Z}^d$ which is an aperiodic translational tile. By applying the "folded bridge" construction above we obtain a set $F' \subset \mathbb{Z}^{d+2}$ which is connected, and is also an aperiodic translational tile (by Theorem 2.1, since S is a rectangle). Let R_{d+2} be the "dented $(d+2)$-dimensional cube" constructed in the proof of [GT22, Lemma 2.2]. Observe that by construction of R_{d+2}, the set $F' + R_{d+2} \subset \mathbb{R}^{d+2}$ is connected if and only if $F' + [0,1]^{d+2} \subset \mathbb{R}^{d+2}$ is connected; thus, since F' is connected in $\mathbb{Z}^{d+2}$ in the sense of Definition 2.1, $F' + R_{d+2}$ is connected

in $\mathbb{R}^{d+2}$. Moreover, the argument in the proof of [GT22, Theorem 2.1] gives that $F' + R_{d+2}$ is aperiodic in $\mathbb{R}^{d+2}$, since F' is aperiodic in $\mathbb{Z}^{d+2}$. Finally, note that $F' + R_{d+2} \subset \mathbb{R}^{d+2}$ is equal to the closure of its interior. Theorem 1.2 now follows, with Ω being $F' + R_{d+2}$. □

3. Connected spectral sets that do not tile

The ultimate goal of this section is to prove Theorem 1.4. We begin with the following general theorem, which shows that certain operations allow to construct, from a given spectral set Ω, other sets that are spectral as well and that preserve the tiling property of the original set Ω.

Theorem 3.1 (Spectrality and tiling preserving operations)**.** *Let Ω be a bounded, measurable set in $\mathbb{R}^d$. Define the finite set*

$$X = \Big\{(v_j, s_j) : j = 0, 1, \dots, n-1\Big\} \subseteq \mathbb{R}^{d+k}$$

where $v_0, \dots, v_{n-1} \in \mathbb{R}^d$ and $s_0, \dots, s_{n-1}$ are n distinct points in $\mathbb{Z}^k$ such that

$$S = \{s_j \colon j = 0, 1, \dots, n-1\}$$

tiles $\mathbb{Z}^k$ by translations. Let $\Omega' = \left(\Omega \times [0,1]^k\right) \oplus X$. Then:

(i) Ω' tiles $\mathbb{R}^{d+1}$ by translations if and only if Ω tiles $\mathbb{R}^d$ by translations.

(ii) If $\Omega \subset \mathbb{R}^d$ and $S + [0,1]^k \subset \mathbb{R}^k$ are spectral, then Ω' is spectral in $\mathbb{R}^{d+k}$.

Proof of Theorem 3.1 (i). If $A \oplus \Omega = \mathbb{R}^d$ then $A' \oplus \Omega' = \mathbb{R}^{d+k}$, where

$$A' = A \times T$$

and $T \subset \mathbb{Z}^d$ is a tiling of $\mathbb{Z}^k$ by S. Conversely, if $\Omega' \oplus A' = \mathbb{R}^{d+k}$ then, by a similar argument as in the proof of Theorem 2.1, the set (2) is a tiling of $\mathbb{R}^d \times \{0\}^k$ by $\Omega \times \{0\}^k$. □

Proof of Theorem 3.1 (ii). Let $\Lambda \subset \mathbb{R}^d$ be a spectrum for Ω and $\Sigma \subset \mathbb{R}^k$ be a spectrum for $S + [0,1]^k$, then the Cartesian product set

$$\Lambda' := \Lambda \times \Sigma = \{(\lambda, \sigma) \in \mathbb{R}^{d+k} \colon \lambda \in \Lambda,\ \sigma \in \Sigma\}$$

defines an orthogonal system

$$E(\Lambda') = \{e^{2\pi i \lambda' \cdot x}\}_{\lambda' \in \Lambda'}$$

in $L^2(\Omega')$. Indeed, let $\tau = (\lambda, \sigma), \tau' = (\lambda', \sigma')$ be distinct points in Λ'. By (1), we need to show that

$$\widehat{\mathbb{1}_{\Omega'}}(\tau' - \tau) = 0. \tag{4}$$

Observe that by the definition of Ω' we have

$$\mathbb{1}_{\Omega'}(w_1, \dots, w_{d+k}) = \mathbb{1}_{\Omega}(w_1, \dots, w_d)\mathbb{1}_{[0,1]^k}(w_{d+1}, \dots, w_{d+k}) * \left(\sum_{x \in X} \delta_x\right)(w_1, \dots, w_{d+k}).$$

Therefore

$$\widehat{\mathbb{1}_{\Omega'}}(\xi_1, \dots, \xi_{d+k}) = \widehat{\mathbb{1}_{\Omega}}(\xi_1, \dots, \xi_d)\widehat{\mathbb{1}_{[0,1]^k}}(\xi_{d+1}, \dots, \xi_{d+k})\left(\sum_{x \in X} e^{2\pi i x \cdot (\xi_1, \dots, \xi_{d+k})}\right). \tag{5}$$

If λ', λ are distinct in Λ, then by (1)

$$\widehat{\mathbb{1}_{\Omega}}(\lambda' - \lambda) = 0,$$

since Λ is a spectrum for Ω, and so, in particular $E(\Lambda)$ is orthogonal in $L^2(\Omega)$. Thus, in this case by (5) we see that (4) is satisfied. Otherwise, $\lambda' - \lambda = 0$ and σ, σ' are distinct in the spectrum Σ of $S + [0,1]^k$, so by (1)

$$\widehat{\mathbb{1}_{S+[0,1]^k}}(\sigma' - \sigma) = 0.$$

By (5) we then have:

$$\begin{aligned}\widehat{\mathbb{1}_{\Omega'}}(\tau' - \tau) &= \widehat{\mathbb{1}_{\Omega}}(0)\widehat{\mathbb{1}_{[0,1]^k}}(\sigma' - \sigma)\left(\sum_{x\in X} e^{2\pi i x\cdot(0,\sigma'-\sigma)}\right)\\ &= |\Omega|\widehat{\mathbb{1}_{[0,1]^k}}(\sigma' - \sigma)\left(\sum_{s\in S} e^{2\pi i s\cdot(\sigma'-\sigma)}\right)\\ &= |\Omega|\widehat{\mathbb{1}_{S+[0,1]^k}}(\sigma' - \sigma) = 0.\end{aligned}$$

Therefore (4) is satisfied in this case as well, and hence $E(\Lambda')$ is orthogonal in $L^2(\Omega')$, as claimed. Now, observe that

$$\operatorname{dens}\Lambda' = \operatorname{dens}\Lambda \times \Sigma = \operatorname{dens}\Lambda \cdot \operatorname{dens}\Sigma.$$

Thus, as Λ is a spectrum for Ω and Σ is a spectrum for $S+[0,1]^k$, by Proposition 1.1, we have

$$\operatorname{dens}\Lambda' = |\Omega||S + [0,1]^k| = |\Omega|n = |\Omega'|.$$

A further application of Proposition 1.1 then gives that Ω' is spectral. □

3.1. **Folded bridge construction in $\mathbb{R}^d$.** Let Ω be a bounded, open set in $\mathbb{R}^d$ with $m+1 > 1$ connected components $C_0, C_1, \ldots, C_m$. Pick $m+1$ points a_j in the interior of C_j each and assume for simplicity $a_0 = 0$. Let K be large enough so that if $\delta_i = \frac{1}{K}(a_{i+1} - a_i)$, $i = 0, \ldots, m-1$, we have

$$C_i \cap (C_i + \delta_j) \neq \emptyset \quad \text{for all } i, j. \tag{6}$$

Let $n = mK + 1$ and define the sequence v_j, $j = 0, 1, 2, \ldots, n-1$, to consist of the n values

$$\begin{aligned}&a_0, a_0 + \delta_0, a_0 + 2\delta_0, \ldots, a_0 + (K-1)\delta_0,\\ &a_1, a_1 + \delta_1, a_1 + 2\delta_1, \ldots, a_1 + (K-1)\delta_1,\\ &a_2, a_2 + \delta_2, a_2 + 2\delta_2, \ldots, a_2 + (K-1)\delta_2,\\ &\qquad\cdots\\ &a_{m-1}, a_{m-1} + \delta_{m-1}, a_{m-1} + 2\delta_{m-1}, \ldots, a_{m-1} + (K-1)\delta_{m-1},\\ &a_m\end{aligned}$$

or:

$$v_j = a_{\tilde{j}} + (j - K\tilde{j})\delta_{\tilde{j}},$$

where $\tilde{j} = \left\lfloor \frac{j}{K} \right\rfloor$, so that, in particular, all points $a_0, a_1, \ldots, a_m$ belong to the sequence v_j, $j = 1, \ldots, n-1$. We then define

$$\Omega_1 = \Omega \times [0,1]^2 + X$$

where

$$\begin{aligned}X &= \{X_0, X_1, \ldots, X_{2n-1}\}\\ &= \{(0, s_0), (0, s_1), (0, s_2), \ldots, (0, s_{n-1}),\\ &\qquad (v_0, s_n), (v_1, s_{n+1}), \ldots, (v_{n-1}, s_{2n-1})\},\end{aligned}$$

where $s_j \in \mathbb{R}^2$ is the sequence defined in (3) and shown in Figure 2. Notice that this is a disjoint sum up to measure zero since the s_j are all different.

Let us now see why the set Ω_1 is connected. The first observation is that the sets $C_j \times [0,1]^2$ are connected. In other words, for every $\omega_1, \omega_2 \in \Omega$ and $x_1, x_2 \in [0,1]^2$

(7) ω_1, ω_2 connected in $\Omega \implies (\omega_1, x_1), (\omega_2, x_2)$ connected[4] in $\Omega \times [0,1]^2$.

When moving from one cell of Figure 2 to the next, the two sets

$$C_i \times [0,1]^2 + (v_j, s_j) \text{ and } C_i \times [0,1]^2 + (v_{j+1}, s_{j+1})$$

are connected to each other because they have a non-empty intersection, by (6) and the fact that s_j and s_{j+1} differ in one coordinate only and exactly by 1, so $s_j - s_{j+1} \in [-1,1]^2$, the latter set being the difference set of $[0,1]^2$.

Hence, when we move across one cell in Figure 2, following the path, the connected components are either maintained or merging, so new connected components are not created along the way. Merging happens when we are moving on the upper row (see an illustration in Figure 3). Take $j \geq 1$ and let v_k be such that $a_j = v_k$. Then

$$(a_j, s_{n+k}) = (a_0 + v_k, s_{n+k}) \in C_0 \times [0,1]^2 + X$$

and

$$(a_j, s_{n-k-1}) \in C_j \times [0,1]^2 + X.$$

At that point the set $C_j \times [0,1]^2 + X$ gets connected to $C_0 \times [0,1]^2 + X$, so, in the end we are left with one connected set.

By Theorem 3.1 we have:

(i) Ω_1 tiles $\mathbb{R}^{d+2}$ by translations if and only if Ω tiles $\mathbb{R}^d$ by translations.

(ii) Ω_1 is spectral in $\mathbb{R}^{d+2}$ if Ω is spectral in $\mathbb{R}^d$.

Using this, we can now prove Theorem 1.4:

Proof of Theorem 1.4. By [T04, Theorem 1.2] and [KM1, Section 3], if $d \geq 3$, we can choose a finite union of closed unit cubes $\Omega \subset \mathbb{R}^d$ which is spectral but does not tile by translations. Hence, by applying the construction above we obtain a set $\Omega_1 \subset \mathbb{R}^{d+2}$ which is connected, spectral and does not tile $\mathbb{R}^{d+2}$ by translations. Moreover, observe that by construction, if Ω is a finite union of closed d-dimensional unit cubes, Ω_1 is a finite union of closed $(d+2)$-dimensional unit cubes. In particular, Ω_1 is the closure of its interior. This proves Theorem 1.4. □

4. Connected translational tiles that are not spectral

Let Ω be a bounded measurable set in $\mathbb{R}^d$ and let $v \in \mathbb{R}^d$ be a vector. Let $u = (v, 1) \in \mathbb{R}^{d+1}$, $n \geq 1$. We say that the set

(8) $$\Omega' := \Omega \times [0,1] + \{0, u, 2u, \ldots, (n-1)u\}$$

is a stacking of Ω. See Figure 4 for a visual illustration of the notion.

Note that by Theorem 3.1 (ii) we have that a stacking Ω' of Ω tiles $\mathbb{R}^{d+1}$ by translations if and only if Ω tiles $\mathbb{R}^d$ by translations.

In addition, we have the following:

Theorem 4.1. *Let Ω be a measurable set in $\mathbb{R}^d$ of finite measure. Suppose that Ω' is a stacking of Ω. If $\Omega' \subset \mathbb{R}^{d+1}$ is spectral then $\Omega \subset \mathbb{R}^d$ is spectral.*

[4] We say that two points are connected in a set if they both belong to the same connected component of the set.

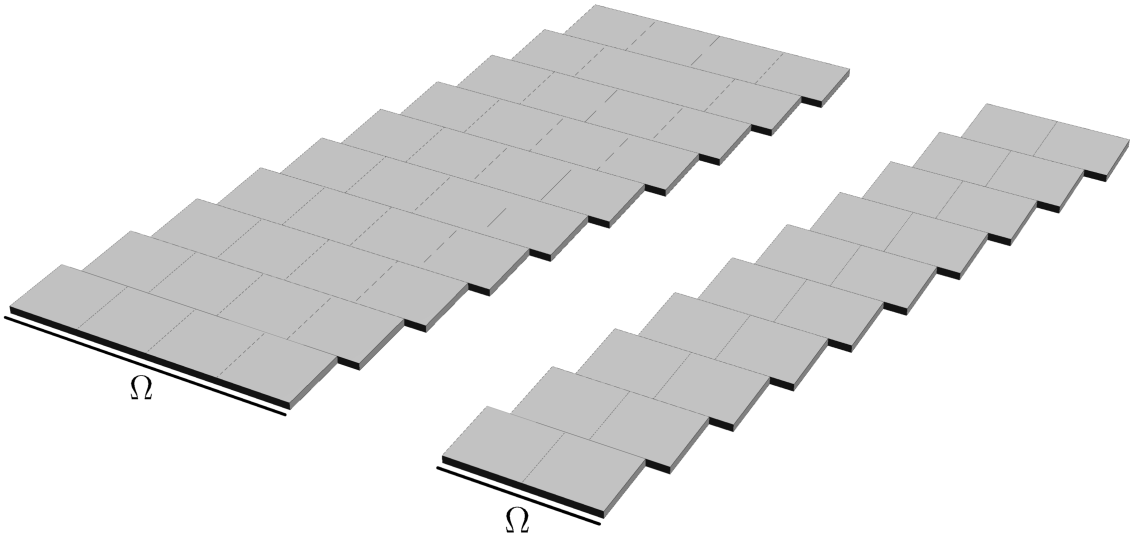

FIGURE 4. A stacking Ω' of the set Ω in one dimension higher.

Remark 4.1. Note that Theorem 3.1(ii) gives that the converse is also true: If $\Omega \subset \mathbb{R}^d$ is spectral then $\Omega' \subset \mathbb{R}^{d+1}$ is spectral. However, in this section we will only use the direction in the statement of Theorem 4.1.

Proof. We have

$$\mathbb{1}_{\Omega'} = \mathbb{1}_{\Omega\times[0,1]} * (\delta_0 + \delta_u + \cdots + \delta_{(n-1)u})$$

so, when $u \cdot \xi \notin \mathbb{Z}$, with $\xi = (\xi_1, \xi_2, \ldots, \xi_{d+1}) \in \mathbb{R}^{d+1}$, we have

$$\widehat{\mathbb{1}_{\Omega'}}(\xi) = \widehat{\mathbb{1}_{\Omega}}(\xi_1, \ldots, \xi_d)\widehat{\mathbb{1}_{[0,1]}}(\xi_{d+1})\left(\sum_{j=0}^{n-1} e^{2\pi i j(u\cdot\xi)}\right)$$

$$= \widehat{\mathbb{1}_{\Omega}}(\xi_1, \ldots, \xi_d)\widehat{\mathbb{1}_{[0,1]}}(\xi_{d+1})\frac{1 - e^{2\pi i n(u\cdot\xi)}}{1 - e^{2\pi i(u\cdot\xi)}}. \tag{9}$$

(Since we care about zeros introduced beyond those of $\widehat{\mathbb{1}_{\Omega\times[0,1]}}$ we may assume that $u \cdot \xi \notin \mathbb{Z}$ – see below.) Define the subgroup of $\mathbb{R}^{d+1}$

$$G = \left\{\xi = (\xi_1, \xi_2, \ldots, \xi_{d+1}) : u \cdot \xi \in \frac{1}{n}\mathbb{Z}\right\}$$

and its subgroup of index n

$$H = \{\xi = (\xi_1, \xi_2, \ldots, \xi_{d+1}) : u \cdot \xi \in \mathbb{Z}\}.$$

From (9) it follows that the zeros of $\widehat{\mathbb{1}_{\Omega'}}$ are those due to $\widehat{\mathbb{1}_{\Omega\times[0,1]}}$ plus the union of cosets of H in G

$$D = \left(H + \frac{u}{n\|u\|^2}\right) \cup \left(H + \frac{2u}{n\|u\|^2}\right) \cup \ldots \cup \left(H + \frac{(n-1)u}{n\|u\|^2}\right).$$

If two distinct points of $\mathbb{R}^{d+1}$ are in the same coset of H then their difference is in H, so it is not in D.

Suppose $\Lambda' \subseteq \mathbb{R}^{d+1}$ is a spectrum of Ω'. Then, by Proposition 1.1:

$$\operatorname{dens}\Lambda' = |\Omega'| = n|\Omega| = n|\Omega \times [0,1]|$$

(since for every $0 \le j < j' \le n-1$, $|(\Omega \times [0,1] + ju) \cap (\Omega \times [0,1] + j'u)| = 0$). We will now select elements of Λ' of density at least $|\Omega \times [0,1]|$ whose pairwise differences do not intersect D. If we call Λ the set of those elements of Λ' that we kept, it follows that the pairwise differences of Λ all fall in $\left\{\widehat{\mathbb{1}_{\Omega\times[0,1]}} = 0\right\}$.

To select the points of Λ', we want we look at every coset $\lambda + G$, $\lambda \in \Lambda'$. For each $\lambda_0 \in \Lambda'$, at least a fraction $1/n$ of the points in $\Lambda' \cap \lambda_0 + G$ are on one of the cosets

$$\lambda_0 + H + j\tilde{u}, \quad j = 0, 1, \dots, n-1 \tag{10}$$

of H, where

$$\tilde{u} = \frac{u}{n\|u\|^2}.$$

We keep precisely those points of Λ' on $\lambda_0 + G$, i.e., those on the most populated (highest density) of the n cosets (10). It follows that for any two points we kept their difference is either not in G (hence also not in $D \subseteq G$) or, if their difference is in G, then it is in H, hence again not in D.

Thus, we conclude that if Λ' is a spectrum for Ω' then we have that $E(\Lambda)$ is orthogonal in $L^2(\Omega \times [0,1])$. Moreover, by the construction of Λ its lower density is bounded from below by

$$\frac{1}{n}\operatorname{dens}\Lambda' = |\Omega \times [0,1]|.$$

By Proposition 1.1, we then have that $\Omega \times [0,1]$ is spectral. Thus, from [GL16, Theorem 1.1] it follows that Ω is spectral. □

Using Theorem 4.1, we can finally prove Theorem 1.5:

Proof of Theorem 1.5. Let $d \geq 3$. By [KM2], we can choose a finite disjoint union of unit cubes $\Omega \subset \mathbb{R}^d$ which tiles the space by translations and is not spectral.

Our goal is to construct higher dimensional bridges between the connected components of Ω while preserving its tiling and spectral properties.

We denote by $C_1, \dots, C_m \subset \Omega$, $m > 1$, the connected components of Ω, and let $\tilde{C}_j$ be the set of the centers of the cubes that C_j consists of. We may assume, without the loss of generality, that

$$\min_{1 \leq i < j \leq m} \min\{\|c_i - c_j\| \colon c_i \in \tilde{C}_i, c_j \in \tilde{C}_j\} = \min\{\|c_1 - c_2\| \colon c_1 \in \tilde{C}_1, c_2 \in \tilde{C}_2\}.$$

Let

$$D_{(C_1,C_2)} := \min\{\|c_1 - c_2\| \colon c_1 \in \tilde{C}_1, c_2 \in \tilde{C}_2\} = \|b - a\|$$

where $a \in \tilde{C}_1$, $b \in \tilde{C}_2$ are centers of unit cubes in C_1, C_2 of minimal distance.

Let $n = \left\lceil D_{(C_1,C_2)} \right\rceil$ be the natural number closest (from above) to $\|b-a\|$, so that

$$|C_j \cap (C_j + v)| > 0, \quad j = 1, \dots, m \tag{11}$$

where $v = \frac{(b-a)}{n} \in \mathbb{R}^d$. Consider the stacking Ω_1 of Ω:

$$\Omega_1 := (\Omega \times [0,1]) \oplus \left\{0, u, 2u, \dots, \left\lfloor \frac{n}{2} \right\rfloor u\right\} \tag{12}$$

where $u = (v, 1) \in \mathbb{R}^{d+1}$. This is a disjoint sum because of the 1 in the last coordinate of u, up to measure zero. In other words, for every $0 \leq j < j' \leq \left\lfloor \frac{n}{2} \right\rfloor$

$$|(\Omega \times [0,1] + ju) \cap (\Omega \times [0,1] + j'u)| = 0.$$

By Theorem 3.1 we know that Ω_1 tiles $\mathbb{R}^{d+1}$ since Ω tiles $\mathbb{R}^d$ and from Theorem 4.1 we also have that Ω_1 is not spectral since Ω is not spectral. We denote

$$C_j^1 = (C_j \times [0,1]) \oplus \left\{0, u, 2u, \dots, \left\lfloor \frac{n}{2} \right\rfloor u\right\}, \quad j = 1, \dots, m.$$

Then, by (11), for each $j = 1, \dots, m$, the set C_j^1 is a connected finite union of unit cubes with centers

$$\tilde{C}_j^1 := \left(\tilde{C}_j \times \left\{\frac{1}{2}\right\}\right) \oplus \left\{0, u, 2u, \dots, \left\lfloor \frac{n}{2} \right\rfloor u\right\}.$$

Let

$$D_{(C_1^1, C_2^1)} := \min\left\{\left\|c_1^1 - c_2^1\right\| : c_1^1 \in \tilde{C}_1^1, c_2^1 \in \tilde{C}_2^1\right\}$$

and $n_1 := \left\lceil D_{(C_1^1, C_2^1)} \right\rceil$. Observe that, as

$$\left(a, \frac{1}{2}\right) + \left\lfloor \frac{n}{2} \right\rfloor u \in C_1^1, \quad \left(b, \frac{1}{2}\right) \in C_2^1,$$

and

$$n - 1 < \|a - b\| = D_{C_1, C_2} \le n$$

we have:

$$\begin{aligned}
D_{(C_1^1, C_2^1)} &\le \left\| \left(a, \frac{1}{2}\right) + \left\lfloor \frac{n}{2} \right\rfloor u - \left(b, \frac{1}{2}\right) \right\| \\
&= \left\| \left((a-b) + \left\lfloor \frac{n}{2} \right\rfloor \frac{(b-a)}{n}, \left\lfloor \frac{n}{2} \right\rfloor \right) \right\| \\
&= \frac{1}{2} \begin{cases} \left\| \left(\frac{n+1}{n}(a-b), n-1 \right) \right\| & n \text{ is odd} \\ \|(a-b, n)\| & n \text{ is even} \end{cases} \\
&= \frac{1}{2} \begin{cases} \sqrt{\left(\frac{n+1}{n}\|a-b\|\right)^2 + (n-1)^2} & n \text{ is odd} \\ \sqrt{\|a-b\|^2 + n^2} & n \text{ is even} \end{cases} \\
&\le \frac{1}{2} \begin{cases} \sqrt{2\left(\frac{n+1}{n}\|a-b\|\right)^2} & n \text{ is odd} \\ \sqrt{2n^2} & n \text{ is even} \end{cases} \\
&= \frac{1}{\sqrt{2}} \begin{cases} \frac{n+1}{n} D_{(C_1, C_2)} & n \text{ is odd} \\ n & n \text{ is even} \end{cases}. \qquad (13)
\end{aligned}$$

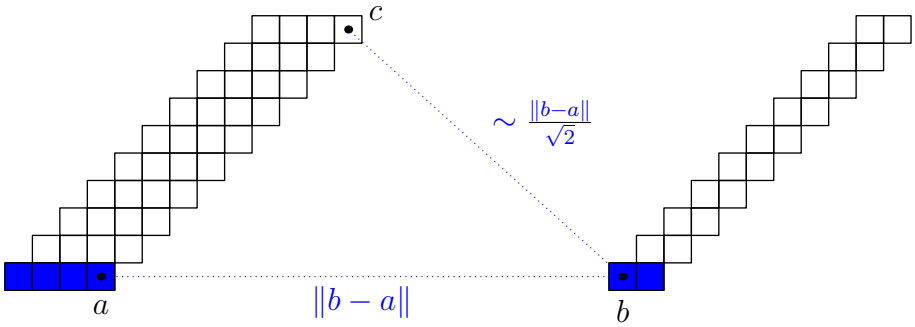

FIGURE 5. Shortening the distance between two connected components. The slope of the line from a to c is approximately 1 when n is large. The blue set is $\Omega \times [0,1] \subseteq \mathbb{R}^d \times \mathbb{R}$.

We have the following possible cases:

Case 1: If $D_{(C_1, C_2)} < 2$, then $n \le 2$ and we have that $C_1^1 \cup C_2^1$ is connected. Indeed, clearly $D_{(C_1, C_2)} > 1$ as otherwise $C_1 \cap C_2$ is non-empty but this contradicts the assumption that C_1, C_2 are different connected components; therefore, we must have $n = 2$, $u = (\frac{b-a}{2}, 1)$ and the cube in C_1^1 that is centered

at $(\frac{a+b}{2}, \frac{3}{2})$ intersects also the side of the cube centered at $(b, \frac{1}{2})$ in C_2^1. Thus, Ω_1 has at most $m-1$ connected components. (See Figure 6.)

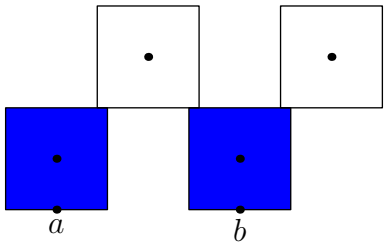

FIGURE 6. This is Case 1, with $D_{(C_1,C_2)} < 2$.

Case 2: If $D_{(C_1,C_2)} \geq 2$, then, by (13):

$$D_{(C_1^1,C_2^1)} < \frac{4}{3\sqrt{2}} D_{(C_1,C_2)} < 0.94281 \cdot D_{(C_1,C_2)}. \tag{14}$$

Indeed, if $D_{(C_1,C_2)} = n = 2$, then (13) gives

$$D_{(C_1^1,C_2^1)} \leq \frac{D_{(C_1,C_2)}}{\sqrt{2}}$$

which implies (14). If $3 \leq n$ is odd, then (13) implies (14), since

$$\frac{(n+1)}{n} \leq \frac{4}{3}$$

in this case. Otherwise, $4 \leq n$ is even, and then by (13) we have

$$D_{(C_1^1,C_2^1)} \leq \frac{n}{\sqrt{2}} \leq \frac{D_{(C_1,C_2)} + 1}{\sqrt{2}},$$

which implies (14) since

$$D_{(C_1,C_2)} + 1 < \frac{4}{3} D_{(C_1,C_2)}$$

in this case.

Unless Case 1 applies, we repeat the process above. In the k-th iteration ($k \geq 2$), the distance between the components C_1^{k-1} and C_2^{k-1} of $\Omega_{k-1} \subset \mathbb{R}^{d+k-1}$ shrinks at a uniform rate in $C_1^k, C_2^k \subset \Omega_k \subset \mathbb{R}^{d+k}$. (See Figure 5.) Hence, after $l < \infty$ iterations, we obtain a set Ω_l in $\mathbb{R}^{d+l}$ which is a tile and is not spectral and such that $D_{(C_1^{l-1},C_2^{l-1})} < 2$. Therefore, as in Case 1 above, the set $C_1^l \cup C_2^l$ in $\Omega_l \subset \mathbb{R}^{d+l}$ is connected. We constructed a "spiral bridge" in $\Omega_l \subset \mathbb{R}^{d+l}$ between the original components C_1 and C_2 of Ω, thus Ω_l has at most $m-1$ connected components.

We iterate this process, constructing $m-1$ spiral bridges between all the components of the original set Ω while preserving its tiling and non-spectrality properties, to eventually obtain a connected set $\tilde{\Omega} \subset \mathbb{R}^{\tilde{d}}$ which tiles the space by translations and is not spectral. Finally, observe that by construction, $\tilde{\Omega}$ is a finite union of closed unit cubes, hence $\tilde{\Omega}$ is the closure of its interior. This completed the proof of Theorem 1.5. □

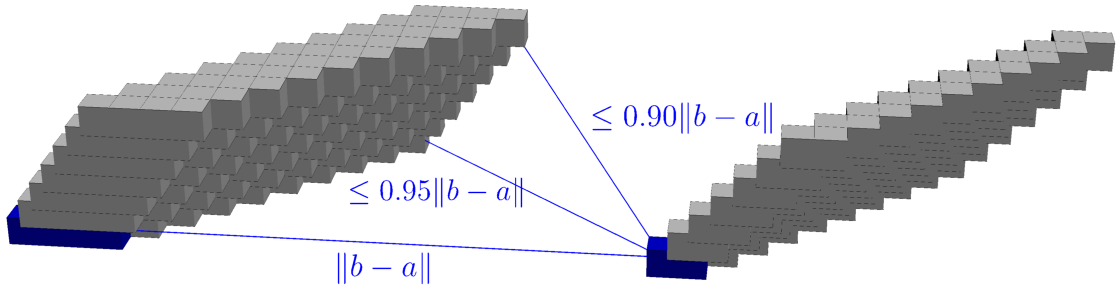

FIGURE 7. Two steps of the stacking procedure. The blue set is $\Omega \times [0,1]^2 \subseteq \mathbb{R}^d \times \mathbb{R}^2$. The distance between the two connected components is being reduced exponentially. The bottom layer of the cubes is the same as that in Fig. 5, where they are shown in dimension $d+1$ (after just one step of the stacking procedure.

5. Discussion and open problems

5.1. **Repairing the periodic tiling conjecture.** Despite the fact that several positive results towards Conjecture 1.1 have been obtained over the years (see [GT22, Section 1] for a partial list), the conjecture was recently proven to be false in high dimensions [GT22]. However, the aperiodic translational tile constructed in [GT22] is a very complicated disconnected set, and, on the other hand, Conjecture 1.1 is known to hold for convex domains in *all* dimensions [M80, V54] in a strong sense: every convex translational tile is also a lattice tile. This naturally motivates one to seek the weakest regularity assumption on the structure of a set under which the periodic tiling conjecture is true in all dimensions.

In this paper we construct aperiodic translational tiles which are connected, showing that a connectedness assumption is not strong enough for the purpose of repairing the periodic tiling conjecture. We therefore must strengthen it, and look for a regularity assumption in the spectrum between connectedness and convexity. This gives rise to the following questions:

Question 1. *Does Conjecture 1.1 hold for simply connected sets in all dimensions?*

We suspect that by adapting the method in this paper, constructing folded bridges between the connected components, one might prove a negative answer to Question 1. Upon a negative answer to Question 1, we can further ask:

Question 2. *Does Conjecture 1.1 hold for topological balls in all dimensions?*

Note that while Conjecture 1.1 is still open in the plane[5], it is known to be true for topological disks [BN91, Ken92].

5.2. **Repairing Fuglede's conjecture.** Conjecture 1.3 inspired extensive research concerning the connection between spectrality and tiling by translations. Over time, it has became apparent that in many respects, spectral sets "behave like" sets which can tile the space by translations. However, after a few decades, counterexamples to both directions of the conjecture were constructed in dimension $d \geq 3$ (see [KM10, Section 4] and the references therein).

[5]But is known to be true in $\mathbb{Z}^2$ [B20].

Although the connection between the analytic notion of spectrality and the geometric notion of tiling by translations has been intensively studied, the precise connection is still a mystery.

Question 3. *What is the precise connection between spectral sets and translational tiles?*

This suggests the problem of determining the exact conditions under which Conjecture 1.3 holds. In this paper, we solve the problem for *connected* sets, showing that there are connected counterexamples to Fuglede's conjecture. On the other hand, Conjecture 1.3 was proven to hold for convex domain in all dimensions [IKT03, GL17, LM19]. This suggests the study of the following question:

Question 4. *Are there any topological conditions on a set that force either of the directions of the Conjecture 1.3 to be true?*

5.3. **Connectedness in low dimensions.** Our main results, Theorems 1.2, 1.4 and 1.5, demonstrate that the higher the dimension is the weaker a connectedness assumption becomes. In particular, we show that any aperiodic d-dimensional translational tile gives rise to a $(d+2)$-dimensional aperiodic *connected* translational tile. One can ask about the necessity of the two additional dimensions, as follows:

Question 5. *What is the minimal d such that there is a d-dimensional connected aperiodic translational tile?*

We can ask the corresponding questions in the context of Conjecture 1.3:

Question 6. *What is the minimal $d \leq 5$ such that there is a d-dimensional connected counterexample to the direction "spectral $\Rightarrow$ tiles" of Conjecture 1.3?*

Question 7. *What is the minimal d such that there is a d-dimensional connected counterexample to the direction "tiles $\Rightarrow$ spectral" of Conjecture 1.3?*

In particular, can the proof of Theorem 1.5 be amended to give a connected, non-spectral tile in a known dimension, in the spirit of Theorem 1.4? Notice that the construction in the proof of Theorem 1.5 of spiral bridges goes up in dimension by a number that depends on the tile we are starting from.

5.4. **Aperiodicity and spectrality.** In [F74] it was observed that by the Poisson summation formula, for a lattice $\Lambda \subset \mathbb{R}^d$, a measurable set $\Omega \subset \mathbb{R}^d$ tiles by translations along Λ if and only if the dual lattice Λ^* is a spectrum for Ω. This might be regarded as the motivation for Conjecture 1.3. Thus, the recent discovery of aperiodic translational tiles [GT22] brings up the question about possible connection between counterexamples to Conjecture 1.1 and counterexamples to Conjecture 1.3:

Question 8. *Is there any aperiodic translational tile $\Omega \subset \mathbb{R}^d$ which is spectral?*

Note that a negative answer to Question 8 would give rise to a new class of counterexamples to Fuglede's conjecture.

5.5. **Quantitative aperiodicity in dimension 1.** It is well known that if a finite $F \subseteq \mathbb{Z}$ tiles $\mathbb{Z}$ by translation then the tiling is necessarily periodic [N77]. In other words if $F \oplus A = \mathbb{Z}$ then there is $N > 0$ such that $A + N = A$. How large can or must this N be compared to a measure of size of F, let us say compared to

the diameter D of F? While it is known that N can be even exponentially large in D [K03, S05, S09] , and must be at most polynomially large in D, when $|F|$ is kept fixed [GT20], no example of a tile F is known where the minimal possible such N (over all possible tilings by F) is more than linearly large in D. Such a tile F, all of whose tilings by translation would have periods much larger than D, would be a one-dimensional, quantitative analogue of aperiodicity.

Question 9. *Does there exist a family of finite sets $F_n \subseteq \mathbb{Z}$ with diameter*

$$\operatorname{diam} F_n \to \infty$$

which tile by translation and the minimal period N_n of the tilings that F_n admits satisfies

$$\frac{N_n}{\operatorname{diam} F_n} \to \infty?$$

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