Turn in your solutions by 4/5/2020. See directions in the class webpage.

1. Let X be the space $C^1([a,b])$ of functions with a continuous derivative in $[a,b]$ (side derivatives at the end-points). We define the norm.

$$\|f\| = |f(a)| + \|f'\|_\infty.$$

Show that this is indeed a norm and that X, with this norm, is a complete space.

💡For $x \in [a,b]$ we have $f(x) = f(a) + \int_a^x f'(t)\,dt$.

2. Consider the sequence space $\ell^1(\mathbb{N})$ which consists of all complex sequences $x = (x_1, x_2, \ldots)$ such that

$$\sum_{n=1}^{\infty} |x_n| < \infty.$$

The norm is $\|x\|_1 = \sum_{n=1}^{\infty} |x_j|$. Consider the operator $T : \ell^1(\mathbb{N}) \to \ell^1(\mathbb{N})$ defined by

$$Tx = (x_2, x_3, \ldots).$$

Show that it is a bounded operator and find its norm.

3. Consider the Banach space $\ell^2(\mathbb{N})$ which consists of all complex sequences $x = (x_1, x_2, \ldots)$ such that

$$\sum_{n=1}^{\infty} |x_n|^2 < \infty.$$

The norm is $\|x\| = \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2}$.

If the series $\sum_{n=1}^{\infty} a_n x_n$ converges for every $x \in \ell^2(\mathbb{N})$ (i) show that

$$Tx = \sum_{n=1}^{\infty} a_n x_n$$

is a bounded linear functional $\ell^2(\mathbb{N}) \to \mathbb{C}$. (ii) Show also that $\sum_{n=1}^{\infty} |a_n|^2 < \infty$ (in other words the sequence $a = (a_1, a_2, \ldots)$ is in $\ell^2(\mathbb{N})$).

💡For the first question apply the Banach-Steinhaus theorem to the sequence of functionals

$$T_N x = \sum_{n=1}^{N} a_n x_n.$$