Turn in your solutions by 31/3/2020. See directions in the class webpage.

**1.** In our lectures about Weyl's theorem on equidistribution (but also in our notes and in the book of Stein and Shakarchi) Weyl's theorem refers to continuous functions which have period 1. This is natural because the trigonometric polynomials of the form

$$\sum_{k=-N}^{N} p_k e^{2\pi i k x}$$

which are used in the proof (but also in the statement of the theorem) approximate uniformly (according to Fejér's theorem) only functions which have period 1.

But this restriction is unnecessary. Show that if for the sequence $a_n \in [0,1)$ we have

$$\frac{1}{N}\sum_{k=1}^{N} f(a_k) \to \int_0^1 f \qquad (1)$$

for every continuous and 1-periodic function $f$ then (1) holds for every continuous, not necessarily periodic, function.

In the proof we gave we showed (1) for every step function, not necessarily periodic.

**2.** If $a \in \mathbb{R} \setminus \{0\}$ and $0 < \sigma < 1$ show that the sequence $\{an^\sigma\}$ is uniformly distributed in $[0,1]$. ($\{x\}$ denotes the fractional part of $x \in \mathbb{R}$.)

Use Weyl's criterion. Approximate the sum $\sum_{n=1}^{N} e^{2\pi i k \{an^\sigma\}} = \sum_{n=1}^{N} e^{2\pi i k a n^\sigma}$ by the integral $\int_1^N e^{2\pi i k a x^\sigma}\,dx$ and bound their difference using the Mean Value Theorem in every interval of the form $[i, i+1]$.

**3.** Working as in Exercise 2 show that the sequence $\{a \log n\}$ is not uniformly distributed in $[0,1]$ for any $a \in \mathbb{R}$.