The Fourier Transform and applications

Mihalis Kolountzakis

University of Crete

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Groups and Haar measure

Locally compact abelian groups:

- Integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- Finite cyclic group $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$: addition $\bmod m$
- Reals $\mathbb{R}$
- Torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$: addition of reals $\bmod 1$
- Products: $\mathbb{Z}^d$, $\mathbb{R}^d$, $\mathbb{T} \times \mathbb{R}$, etc

Haar measure on G = translation invariant on G: $\mu(A) = \mu(A + t)$.
Unique up to scalar multiple.

- Counting measure on $\mathbb{Z}$
- Counting measure on $\mathbb{Z}_m$, normalized to total measure 1 (usually)
- Lebesgue measure on $\mathbb{R}$
- Lebesgue masure on $\mathbb{T}$ viewed as a circle
- Product of Haar measures on the components

Characters and the dual group

- Character is a (continuous) group homomorphism from G to the *multiplicative* group $U = \{z \in \mathbb{C} : |z| = 1\}$.
- $\chi : G \to U$ satsifies $\chi(h+g) = \chi(h)\chi(g)$
- If χ, ψ are characters then so is $\chi\psi$ (pointwise product). Write $\chi + \psi$ from now on instead of $\chi\psi$.
- Group of characters (written *additively*) $\widehat{G}$ is the dual group of G
- $G = \mathbb{Z} \Longrightarrow \widehat{G} = \mathbb{T}$: the functions $\chi_x(n) = \exp(2\pi i x n), x \in \mathbb{T}$
- $G = \mathbb{T} \Longrightarrow \widehat{G} = \mathbb{Z}$: the functions $\chi_n(x) = \exp(2\pi i n x), n \in \mathbb{Z}$
- $G = \mathbb{R} \Longrightarrow \widehat{G} = \mathbb{R}$: the functions $\chi_t(x) = \exp(2\pi i t x), t \in \mathbb{R}$
- $G = \mathbb{Z}_m \Longrightarrow \widehat{G} = \mathbb{Z}_m$: the functions $\chi_k(n) = \exp(2\pi i k n/m), k \in \mathbb{Z}_m$
- $G = A \times B \Longrightarrow \widehat{G} = \widehat{A} \times \widehat{B}$
- Example: $G = \mathbb{T} \times \mathbb{R} \Longrightarrow \widehat{G} = \mathbb{Z} \times \mathbb{R}$. The characters are $\chi_{n,t}(x,y) = \exp(2\pi i(nx + ty))$.
- G is compact $\Longleftrightarrow$ $\widehat{G}$ is discrete
- PONTRYAGIN duality: $\widehat{\widehat{G}} = G$.

The Fourier Transform of integrable functions

- $f \in L^1(G)$. That is $\|f\|_1 := \int_G |f(x)|\, d\mu(x) < \infty$
- If G is finite then $L^1(G)$ is all functions $G \to \mathbb{C}$
- The FT of f is $\widehat{f} : \widehat{G} \to \mathbb{C}$ defined by

$$\widehat{f}(\chi) = \int_G f(x)\overline{\chi(x)}\, d\mu(x), \quad \chi \in \widehat{G}$$

- Example: $G = \mathbb{T}$ ("Fourier coefficients"):

$$\widehat{f}(n) = \int_{\mathbb{T}} f(x) e^{-2\pi i n x}\, dx, \quad n \in \mathbb{Z}$$

- Example: $G = \mathbb{R}$ ("Fourier transform"):

$$\widehat{f}(\xi) = \int_{\mathbb{T}} f(x) e^{-2\pi i \xi x}\, dx, \quad \xi \in \mathbb{R}$$

- Example: $G = \mathbb{Z}_m$ ("Discrete Fourier transform or DFT"):

$$\widehat{f}(k) = \frac{1}{m} \sum_{j=0}^{m-1} f(j) e^{-2\pi i k j/m}, \quad k \in \mathbb{Z}_m$$

Elementary properties of the Fourier Transform

- Linearity: $\widehat{\lambda f + \mu g} = \lambda \widehat{f} + \mu \widehat{g}$.
- Symmetry: $\widehat{f}(-x) = \overline{\widehat{\overline{f}}(x)}$, $\widehat{\overline{f}}(x) = \overline{\widehat{f}(-x)}$
- Real f: then $\widehat{f}(x) = \overline{\widehat{f}(-x)}$
- Translation: if $\tau \in G, \xi \in \widehat{G}$, $f_\tau(x) = f(x-\tau)$ then $\widehat{f_\tau}(\xi) = \overline{\xi(\tau)} \cdot \widehat{f}(\xi)$.
  Example: $G = \mathbb{T}$: $f\widehat{(x-\theta)}(n) = e^{-2\pi i n\theta}\widehat{f}(n)$, for $\theta \in \mathbb{T}, n \in \mathbb{Z}$.
- Modulation: If $\chi, \xi \in \widehat{G}$ then $\chi\widehat{(x) f(x)}(\xi) = \widehat{f}(\xi - \chi)$.
  Example: $G = \mathbb{R}$: $e^{\widehat{2\pi i t x} f(x)}(\xi) = \widehat{f}(\xi - t)$.
- $f, g \in L^1(G)$: their convolution is $f * g(x) = \int_G f(t) g(x-t)\, d\mu(t)$.
  Then $\|f * g\|_1 \le \|f\|_1 \|g\|_1$ and

$$\widehat{f * g}(\xi) = \widehat{f}(\xi) \cdot \widehat{g}(\xi), \quad \xi \in \widehat{G}$$

Orthogonality of characters on compact groups

- If G is compact ($\Longrightarrow$ total Haar measure $= 1$) then characters are in $L^1(G)$, being bounded.
- If $\chi \in \widehat{G}$ then

$$\int_G \chi(x)\,dx = \int_G \chi(x+g)\,dx = \chi(g)\int_G \chi(x)\,dx,$$

  so $\int_G \chi = 0$ if χ nontrivial, 1 if χ is trivial ($= 1$).
- If $\chi, \psi \in \widehat{G}$ then $\chi(x)\psi(-x)$ is also a character. Hence

$$\langle \chi, \psi \rangle = \int_G \chi(x)\overline{\psi(x)}\,dx = \int_G \chi(x)\psi(-x)\,dx = \begin{cases} 1 & \chi = \psi \\ 0 & \chi \neq \psi \end{cases}$$

- Fourier representation (inversion) in $\mathbb{Z}_m$: $G = \mathbb{Z}_m \Longrightarrow$ the m characters form a complete orthonormal set in $L^2(G)$:

$$f(x) = \sum_{k=0}^{m-1} \langle f(\cdot), e^{2\pi i k \cdot} \rangle e^{2\pi i k x} = \sum_{k=0}^{m-1} \widehat{f}(k) e^{2\pi i k x}$$

L^2 of compact G

- Trigonometric polynomials = finite linear combinations of characters on G
- Example: $G = \mathbb{T}$. Trig. polynomials are of the type $\sum_{k=-N}^{N} c_k e^{2\pi i k x}$. The least such N is called the degree of the polynomial.
- Example: $G = \mathbb{R}$. Trig. polynomials are of the type $\sum_{k=1}^{K} c_k e^{2\pi i \lambda_k x}$, where $\lambda_j \in \mathbb{R}$.
- Compact G: Stone - Weierstrass Theorem $\Longrightarrow$ trig. polynomials dense in $C(G)$ (in $\|\cdot\|_\infty$).
- Fourier representation in $L^2(G)$: Compact G: The characters form a complete ONS. Since $C(G)$ is dense in $L^2(G)$:

$$f = \int_{\chi \in \widehat{G}} \widehat{f}(\chi)\chi \, d\chi \quad \text{all } f \in L^2(G), \text{ convergence in } L^2(G)$$

- $\widehat{G}$ necessarily discrete in this case

L^2 of compact G, continued

- Compact G: Parseval formula:

$$\int_G f(x)\overline{g(x)}\,dx = \int_{\widehat{G}} \widehat{f}(\chi)\overline{\widehat{g}(\chi)}\,d\chi.$$

- Compact G: $f \to \widehat{f}$ is an *isometry* from $L^2(G)$ onto $L^2(\widehat{G})$.
- Example: $G = \mathbb{T}$

$$\int_{\mathbb{T}} f(x)\overline{g(x)}\,dx = \sum_{k\in\mathbb{Z}} \widehat{f}(k)\overline{\widehat{g}(k)}, \quad f, g \in L^2(\mathbb{T}).$$

- Example: $G = \mathbb{Z}_m$

$$\sum_{j=0}^{m-1} f(j)\overline{g(j)} = \sum_{k=0}^{m-1} \widehat{f}(k)\overline{\widehat{g}(k)}, \quad \text{all } f, g : \mathbb{Z}_m \to \mathbb{C}$$

Triple correlations in $\mathbb{Z}_p$: an application

- Problem of significance in (a) crystallography, (b) astrophysics: determine a subset $E \subseteq \mathbb{Z}_n$ from its triple correlation:

$$\begin{aligned} N_E(a,b) &= \#\{x \in \mathbb{Z}_n : x, x+a, x+b \in E\}, \quad a,b \in \mathbb{Z}_n \\ &= \sum_{x \in \mathbb{Z}_n} \mathbf{1}_E(x)\mathbf{1}_E(x+a)\mathbf{1}_E(x+b) \end{aligned}$$

  Counts number of occurences of translated 3-point patterns $\{0, a, b\}$.
- E can only be determined up to translation: E and $E+t$ have the same $N(\cdot,\cdot)$.
- For general n it has been proved that $N(\cdot,\cdot)$ cannot determine E even up to translation (non-trivial).
- Special case: E can be determined up to translation from $N(\cdot,\cdot)$ if $n = p$ is a prime.
- Fourier transform of $N_E : \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{R}$ is easily computed:

$$\widehat{N_E}(\xi,\eta) = \widehat{\mathbf{1}_E}(\xi)\widehat{\mathbf{1}_E}(\eta)\widehat{\mathbf{1}_E}(-(\xi+\eta)), \quad \xi,\eta \in \mathbb{Z}_n.$$

Triple correlations in $\mathbb{Z}_p$: an application (continued)

- If $N_E \equiv N_F$ for $E, F \subseteq \mathbb{Z}_n$ then

$$\widehat{\mathbf{1}_E}(\xi)\widehat{\mathbf{1}_E}(\eta)\widehat{\mathbf{1}_E}(-(\xi+\eta)) = \widehat{\mathbf{1}_F}(\xi)\widehat{\mathbf{1}_F}(\eta)\widehat{\mathbf{1}_F}(-(\xi+\eta)), \quad \xi, \eta \in \mathbb{Z}_n \quad (1)$$

- Setting $\xi = \eta = 0$ we deduce $\#E = \#F$.
- Setting $\eta = 0$, and using $\widehat{f}(-x) = \overline{\widehat{f}(x)}$ for real f, we get $\left|\widehat{\mathbf{1}_E}\right| \equiv \left|\widehat{\mathbf{1}_F}\right|$.
- If $\widehat{\mathbf{1}_F}$ is never 0 we divide (1) by its RHS to get

$$\phi(\xi)\phi(\eta) = \phi(\xi+\eta), \quad \text{where } \phi = \widehat{\mathbf{1}_E}/\widehat{\mathbf{1}_F} \quad (2)$$

- Hence $\phi : \mathbb{Z}_n \to \mathbb{C}$ is a character and $\widehat{\mathbf{1}_E} \equiv \phi\widehat{\mathbf{1}_F}$.
- Since $\widehat{\mathbb{Z}_n} = \mathbb{Z}_n$ we have $\phi(\xi) = e^{2\pi i t\xi/n}$ for some $t \in \mathbb{Z}_n$
- Hence $E = F + t$
- So N_E determines E up to translation if $\widehat{\mathbf{1}_E}$ is never 0

Triple correlations in $\mathbb{Z}_p$: an application (conclusion)

- Suppose $n = p$ is a prime, $E \subseteq \mathbb{Z}_p$. Then

$$\widehat{\mathbf{1}_E}(\xi) = \frac{1}{p}\sum_{s\in E}(\zeta^\xi)^s, \quad \zeta = e^{-2\pi i/p} \text{ is a } p\text{-root of unity.} \tag{3}$$

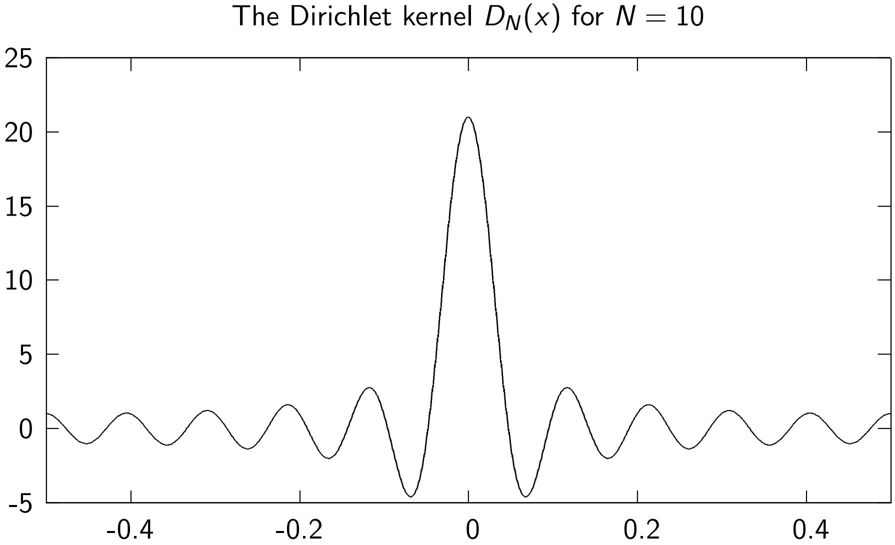
- Each ζ^ξ, $\xi \neq 0$, is a primitive p-th root of unity itself.
- All powers $(\zeta^\xi)^s$ are distinct, so $\widehat{\mathbf{1}_E}(\xi)$ is a subset sum of all primitive p-th roots of unity ($\xi \neq 0$).
- The polynomial $1 + x + x^2 + \cdots + x^{p-1}$ is the *minimal polynomial* over $\mathbb{Q}$ of each primitive root of unity (there are $p - 1$ of them).
- It divides any polynomial in $\mathbb{Q}[x]$ which vanishes on some primitive p-th root of unity
- The only subset sums of all roots of unity which vanish are the empty and the full sum ($E =$ or $E = \mathbb{Z}_p$).
- So in $\mathbb{Z}_p$ the triple correlation $N_E(\cdot,\cdot)$ determines E up to translation.

The basics of the FT on the torus (circle) $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$

- $1 \le p \le q \Longleftrightarrow L^q(\mathbb{T}) \subseteq L^p(\mathbb{T})$: nested L^p spaces. True on compact groups.
- $f \in L^1(\mathbb{T})$: we write $f(x) \sim \sum_{k=-\infty}^{\infty} \widehat{f}(k) e^{2\pi i k x}$ to denote the <u>Fourier series</u> of f. No claim of convergence is made.
- The Fourier coefficients of $f(x) = e^{2\pi i k x}$ is the sequence $\widehat{f}(n) = \delta_{k,n}$.
- The Fourier series of a trig. poly. $f(x) = \sum_{k=-N}^{N} a_k e^{2\pi i k x}$ is the sequence $\ldots, 0, 0, a_{-N}, a_{-N+1}, \ldots, a_0, \ldots, a_N, 0, 0, \ldots$.
- Symmetric partial sums of the Fourier series of f: $S_N(f;x) = \sum_{k=-N}^{N} \widehat{f}(k) e^{2\pi i k x}$
- From $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$ we get easily $S_N(f;x) = f(x) * D_N(x)$, where

$$D_N(x) = \sum_{k=-N}^{N} e^{2\pi i k x} = \frac{\sin 2\pi(N+\frac{1}{2})x}{\sin \pi x}$$ (<u>Dirichlet kernel</u> of order N)

The DIRICHLET kernel



Pointwise convergence

- Important: $\|D_N\|_1 \geq C \log N$, as $N \to \infty$
- $T_N : f \to S_N(f;x) = D_N * f(x)$ is a (continuous) linear functional $C(\mathbb{T}) \to \mathbb{C}$. From the inequality $\|D_N * f\|_\infty \leq \|D_N\|_1 \|f\|_\infty$
- $\|T_N\| = \|D_N\|_1$ is unbounded
- Banach-Steinhaus (uniform boundedness principle) $\Longrightarrow$ Given x there are many continuous functions f such that $T_N(f)$ is unbounded
- Consequence: In general $S_N(f;x)$ does not converge pointwise to $f(x)$, even for continuous f

Summability

- Look at the arithmetical means of $S_N(f;x)$

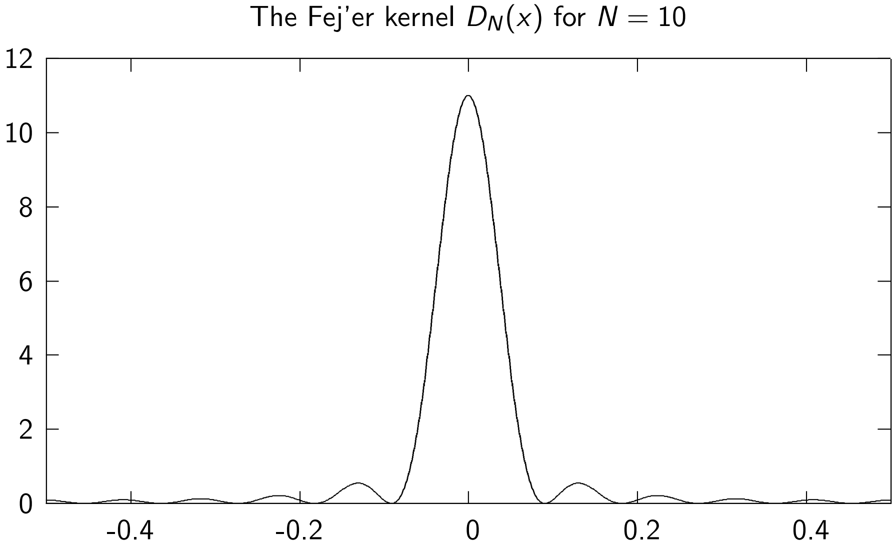
$$\sigma_N(f;x) = \frac{1}{N+1}\sum_{n=0}^{N} S_n(f;x) = K_N * f(x)$$

- The FEJÉR kernel $K_N(x)$ is the mean of the DIRICHLET kernels

$$K_N(x) = \sum_{n=-N}^{N}\left(1-\frac{|n|}{N+1}\right)e^{2\pi i n x} = \frac{1}{N+1}\left(\frac{\sin\pi(N+1)x}{\sin\pi x}\right)^2 \ge 0.$$

- $K_N(x)$ is an approximate identity:
  (a) $\int_{\mathbb{T}} K_N(x)\,dx = \widehat{K_N}(0) = 1$,
  (b) $\|K_N\|_1$ is bounded ($\|K_N\|_1 = 1$, from nonnegativity and (a)),
  (c) for any $\epsilon > 0$ we have $\int_{|x|>\epsilon} |K_N(x)|\,dx \to 0$, as $N \to \infty$

The FEJÉR kernel



Summability (continued)

- K_N approximate identity $\Longrightarrow K_N * f(x) \to f(x)$, in some Banach spaces. These can be:
- $C(\mathbb{T})$ normed with $\|\cdot\|_\infty$: If $f \in C(\mathbb{T})$ then $\sigma_N(f;x) \to f(x)$ uniformly in $\mathbb{T}$.
- $L^p(\mathbb{T})$, $1 \le p < \infty$: If $f \in L^p(\mathbb{T})$ then $\|\sigma_N(f;x) - f(x)\|_p \to 0$
- $C^n(\mathbb{T})$, all n-times C-differentiable functions, normed with $\|f\|_{C^n} = \sum_{k=0}^n \left\|f^{(k)}\right\|_\infty$
- Summability implies uniqueness: the Fourier series of $f \in L^1(\mathbb{T})$ determines the function.
- Another consequence: trig. polynomials are dense in $L^p(\mathbb{T}), C(\mathbb{T}), C^n(\mathbb{T})$
- Another important summability kernel: the Poisson kernel

$$P(r,x) = \sum_{k\in\mathbb{Z}} r^k e^{2\pi i k x}, \quad 0 < r < 1\text{: absolute convergence obvious}$$

Significant for the theory of analytic functions.

The decay of the Fourier coefficients at ∞

- Obvious: $\widehat{f}(n) \le \|f\|_1$
- RIEMANN-LEBESGUE Lemma: $\lim_{|n|\to\infty} \widehat{f}(n) = 0$ if $f \in L^1(\mathbb{T})$. Obviously true for trig. polynomials and they are dense in $L^1(\mathbb{T})$.
- Can go to 0 arbitrarily slowly if we only assume $f \in L^1$.
- $f(x) = \int_0^x g(t)\,dt$, where $\int g = 0$: $\widehat{f}(n) = \frac{1}{2\pi i n}\widehat{g}(n)$ (Fubini)
- Previous implies: $\widehat{f}(|n|) = -\widehat{f}(-|n|) \ge 0 \Longrightarrow \sum_{n\neq 0} \widehat{f}(n)/n < \infty$.
- $\sum_{n>0} \frac{\sin nx}{\log n}$ is not a Fourier series.
- f is an integral $\Longrightarrow \widehat{f}(n) = o(1/n)$: the "smoother" f is the better decay for the FT of f
- $f \in C^2(\mathbb{T}) \Longrightarrow$ absolute convergence for the Fourier Series of f.
- Another condition that imposes "decay":
  $f \in L^2(\mathbb{T}) \Longrightarrow \sum_n \left|\widehat{f}(n)\right|^2 < \infty$.

Interpolation of operators

- T is bounded linear operator on dense subsets of L^{p_1} and L^{p_2}:

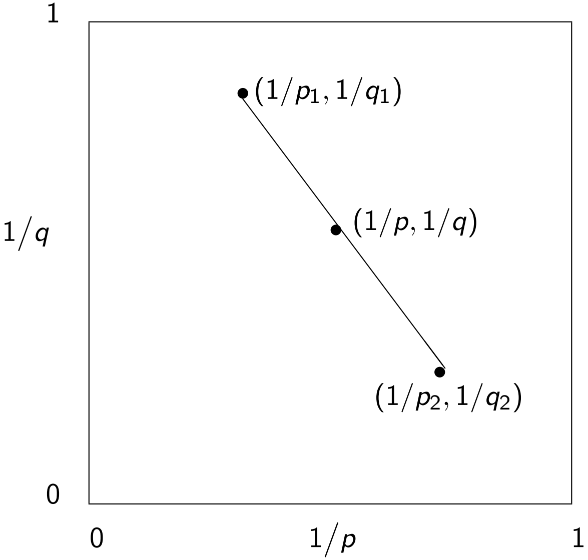
$$\|Tf\|_{q_1} \le C_1 \|f\|_{p_1}, \quad \|Tf\|_{q_2} \le C_2 \|f\|_{p_2}$$

- RIESZ-THORIN interpolation theorem: $T : L^p \to L^q$ for any p between p_1, p_2 (all p's and q's ≥ 1).
- p and q are related by:

$$\frac{1}{p} = t\frac{1}{p_1} + (1-t)\frac{1}{p_2} \implies \frac{1}{q} = t\frac{1}{q_1} + (1-t)\frac{1}{q_2}$$

- $\|T\|_{L^p \to L^q} \le C_1^t C_2^{(1-t)}$
- The exponents $p, q, \ldots$ are allowed to be ∞.

Interpolation of operators: the $1/p, 1/q$ plane



The Hausdorff-Young inequality

- HAUSDORFF-YOUNG: Suppose $1 \le p \le 2$, $\frac{1}{p} + \frac{1}{q} = 1$, and $f \in L^p(\mathbb{T})$. It follows that

$$\left\|\widehat{f}\right\|_{L^q(\mathbb{Z})} \le C_p \|f\|_{L^p(\mathbb{T})}$$

- False if $p > 2$.
- Clearly true if $p = 1$ (trivial) or $p = 2$ (Parseval).
- Use RIESZ-THORIN interpolation for $1 < p < 2$ for the operator $f \to \widehat{f}$ from $L^p(\mathbb{T}) \to L^q(\mathbb{Z})$.

An application: the isoperimetric inequality

- Suppose Γ is a simple closed curve in the plane with perimeter L enclosing area A.

$$A \le \frac{1}{4\pi} L^2 \quad \text{(isoperimetric inequality)}$$

  Equality holds only when Γ is a circle.
- WIRTINGER's inequality: if $f \in C^\infty(\mathbb{T})$ then

$$\int_0^1 \left| f(x) - \widehat{f}(0) \right|^2 dx \le \frac{1}{4\pi^2} \int_0^1 \left| f'(x) \right|^2 dx. \qquad (4)$$

- By smoothness $f(x)$ equals its Fourier series and so does $f'(x) = 2\pi i \sum_n n\widehat{f}(n) e^{2\pi i n x}$
- FT is an isometry (Parseval) so LHS of (4) is $\sum_{n\neq 0} \left|\widehat{f}(n)\right|^2$ while the RHS is $\sum_{n \neq 0} n \left|\widehat{f}(n)\right|^2$ so (4) holds.
- Equality in (4) precisely when $f(x) = \widehat{f}(-1)e^{-2\pi i x} + \widehat{f}(0) + \widehat{f}(1)e^{2\pi i x}$.

An application: the isoperimetric inequality (continued)

- HURWITZ' proof. First assume Γ is smooth, has $L = 1$.
- Parametrization of Γ: $(x(s), y(s))$, $0 \le s \le 1$ w.r.t. arc length s
- $x, y \in C^\infty(\mathbb{T})$, $\quad (x'(s))^2 + (y'(s))^2 = 1$.
- GREEN's Theorem $\Longrightarrow$ area $A = \int_0^1 x(s) y'(s)\, ds$:

$$
\begin{aligned}
A &= \int (x(s) - \widehat{x}(0)) y'(s) = \\
&= \frac{1}{4\pi} \int (2\pi(x(s) - \widehat{x}(0)))^2 + y'(s)^2 - (2\pi(x(s) - \widehat{x}(0)) - y'(s))^2 \\
&\le 1/4\pi \int 4\pi^2 (x(s) - \widehat{x}(0))^2 + y'(s)^2 \quad \text{(drop last term)} \\
&\le 1/4\pi \int x'(s)^2 + y'(s)^2 \quad \text{(WIRTINGER's ineq)} \\
&= 1/4\pi
\end{aligned}
$$

- For equality must have $x(s) = a \cos 2\pi s + b \sin 2\pi s + c$, $y'(s) = 2\pi(x(s) - \widehat{x}(0))$. So $x(s)^2 + y(s)^2$ constant if $c = 0$.

Fourier transform on $\mathbb{R}^n$

- Initially defined only for $f \in L^1(\mathbb{R}^n)$. $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x}\, dx$.
  Follows: $\left\|\widehat{f}\right\|_\infty \le \|f\|_1$. $\widehat{f}$ is continuous.
- Trig. polynomials are not dense anymore in the usual spaces.
- But RIEMANN-LEBESGUE is true. First for indicator function of an interval

  $$[a_1, b_1] \times \cdots \times [a_n, b_n].$$

  Then approximate an L^1 function by finite linear combinations of such.
- Multi-index notation $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$:
  (a) $|\alpha| = \alpha_1 + \cdots + \alpha_n$.
  (b) $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$
  (c) $\partial^\alpha = (\partial/\partial_1)^{\alpha_1} \cdots (\partial/\partial_n)^{\alpha_n}$
- Diff operators $D^j \phi := \frac{1}{2\pi i}(\partial/\partial x_j)$, $D^\alpha \phi = (1/2\pi i)^{|\alpha|} \partial^\alpha$.

- $L^p(\mathbb{R}^n)$ spaces are not nested.
- Not clear how to define $\widehat{f}$ for $f \in L^2$.
- SCHWARTZ class $\mathcal{S}$: those $\phi \in C^\infty(\mathbb{R}^n)$ s.t. for all multiindices α, γ

$$\|\phi\|_{\alpha,\gamma} := \sup_{x \in \mathbb{R}^n} |x^\gamma \partial^\alpha \phi(x)| < \infty.$$

- The $\|\phi\|_{\alpha,\gamma}$ are *seminorms*. They determine the topology of $\mathcal{S}$.
- $C_0^\infty(\mathbb{R}^n) \subseteq \mathcal{S}$
- Easy to see that $\widehat{D^j(\phi)}(\xi) = \xi_j \widehat{\phi}(\xi)$ and $\widehat{x_j\phi(x)}(\xi) = -D^j\widehat{\phi}(\xi)$.
  More generally $\xi^\alpha D^\gamma \widehat{\phi}(\xi) = D^\alpha \widehat{(-x)^\gamma \phi(x)}(\xi)$.
- $\phi \in \mathcal{S} \implies \widehat{\phi} \in \mathcal{S}$ (smoothness $\implies$ decay, decay $\implies$ smoothness)
- Fourier inversion formula: $\phi(x) = \int \widehat{\phi}(\xi) e^{2\pi i \xi x}\, d\xi$.
  Can also write as $\widehat{\widehat{\phi}}(-x) = \phi(x)$.
- We first show its validity for $\phi \in \mathcal{S}$.

Fourier inversion formula on $\mathcal{S}$

- $\widehat{\ }$ shifting: $f, g \in L^1(\mathbb{R}^n) \Longrightarrow \int f\widehat{g} = \int \widehat{f} g$ (Fubini)
- Define the GAUSSIAN function $g(x) = (2\pi)^{-n/2} e^{-|x|^2/2}$, $x \in \mathbb{R}^n$. This normalization gives $\int g(x) = \int |x|^2 g(x) = 1$.
- Using CAUCHY's integral formula for analytic functions we prove $\widehat{g}(\xi) = (2\pi)^{n/2} g(2\pi\xi)$. The Fourier inversion formula holds.
- Write $g_\epsilon(x) = \epsilon^{-n} g(x/\epsilon)$, an approximate identity.
- We have $\widehat{g_\epsilon}(\xi) = (2\pi)^{n/2} g(2\pi\epsilon\xi)$, $\lim_{\epsilon\to 0} \widehat{g_\epsilon}(\xi) = 1$.
- $\widehat{\widehat{\phi}}(-x) = \int e^{2\pi i\xi x} \widehat{\phi}(\xi)\, d\xi = \lim_{\epsilon\to 0} \int e^{2\pi i \xi x} \widehat{\phi}(\xi) \widehat{g_\epsilon}(\xi)\, d\xi$ (dom. conv.)
- $= \lim_{\epsilon\to 0} \int \widehat{\phi(\cdot + x)}(\xi) \widehat{g_\epsilon}(\xi)\, d\xi$ (FT of translation)
- $= \lim_{\epsilon\to 0} \int \phi(x+y) \widehat{\widehat{g_\epsilon}}(y)\, dy$ ($\widehat{\ }$ shifting)
- $= \lim_{\epsilon\to 0} \int \phi(x+y) g_\epsilon(-y)\, dy$ (FT inversion for g_ϵ).
- $= \phi(x)$ (g_ϵ is an approximate identity)

FT on $L^2(\mathbb{R}^n)$

- Preservation of inner product: $\int \phi\overline{\psi} = \int \widehat{\phi}\overline{\widehat{\psi}}$, for $\phi, \psi \in \mathcal{S}$

  Fourier inversion implies $\widehat{\widehat{\overline{\widehat{\psi}}}} = \overline{\psi}$. Use $\widehat{\ }$ shifting.
- PARSEVAL: $f \in \mathcal{S}$: $\|f\|_2 = \left\|\widehat{f}\right\|_2$.
- $\mathcal{S}$ dense in $L^2(\mathbb{R}^n)$: FT extends to L^2 and $f \to \widehat{f}$ is an isometry on L^2.
- By interpolation FT is defined on L^p, $1 \le p \le 2$, and satisfies the HAUSDORFF-YOUNG inequality:

$$\left\|\widehat{f}\right\|_q \le C_p \|f\|_p, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Tempered distributions

- Tempered distributions: $\mathcal{S}'$ is the space of continuous linear functionals on $\mathcal{S}$.
- FT defined on $\mathcal{S}'$: for $u \in \mathcal{S}'$ we define $\widehat{u}(\phi) = u(\widehat{\phi})$, for $\phi \in \mathcal{S}$.
- Fourier inversion for $\mathcal{S}'$: $u(\phi(x)) = \widehat{\widehat{u}}(\phi(-x))$.
- $u \to \widehat{u}$ is an isomorphism on $\mathcal{S}'$
- $1 \le p \le \infty$: $L^p \subseteq \mathcal{S}'$. If $f \in L^p$ this mapping is in $\mathcal{S}'$:

$$\phi \to \int f\phi$$

  Also $\mathcal{S} \subseteq \mathcal{S}'$.
- Tempered measures: $\int (1+|x|)^{-k}\, d|\mu|(x) < \infty$, for some $k \in \mathbb{N}$. These are in $\mathcal{S}'$.
- Differentiation defined as: $(\partial^\alpha u)(\phi) = (-1)^{|\alpha|} u(\partial^\alpha \phi)$.
- FT of L^p functions or tempered measures defined in $\mathcal{S}'$

Examples of tempered distributions and their FT

- $u = \delta_0$, $\widehat{u} = 1$
- $u = D^\alpha \delta_0$. To find its FT

$$\widehat{D^\alpha \delta_0}(\phi) = (D^\alpha \delta_0)(\widehat{\phi}) = (-1)^{|\alpha|} \delta_0(D^\alpha \widehat{\phi}) = (-1)^{|\alpha|} \delta_0(\widehat{(-x)^\alpha \phi}(x))$$

$$= (-1)^{|\alpha|} \widehat{\delta_0}((-x)^\alpha \phi(x)) = \int x^\alpha \phi(x)$$

- So $\widehat{D^\alpha \delta_0} = x^\alpha$.
- $u = x^\alpha$, $\widehat{u} = D^\alpha \delta_0$
- $u = \sum_{j=1}^J a_j \delta_{p_j}$, $\widehat{u}(\xi) = \sum_{j=1}^J a_j e^{2\pi i p_j \xi}$.
- Poisson Summation Formula (PSF): $u = \sum_{k \in \mathbb{Z}^n} \delta_k$, $\widehat{u} = u$

Proof of the POISSON Summation Formula

- $\phi \in \mathcal{S}$. Define $g(x) = \sum_{k\in\mathbb{Z}^n} \phi(x+k)$.
- g has $\mathbb{Z}^n$ as a period lattice: $g(x+k) = g(x)$, $x \in \mathbb{R}^n, k \in \mathbb{Z}^n$.
- The periodization g may be viewed as $g : \mathbb{T}^n \to \mathbb{C}$.
- FT of g lives on $\widehat{\mathbb{T}^n} = \mathbb{Z}^n$. The Fourier coefficients are

$$\widehat{g}(k) = \int_{\mathbb{T}^n} \sum_{m\in\mathbb{Z}^n} \phi(x+m) e^{-2\pi i k x}\, dx = \widehat{\phi}(k).$$

- From decay of $\widehat{\phi}$ follows that the FS of $g(x)$ converges absolutely and uniformly and

$$g(x) = \sum_{k\in\mathbb{Z}^n} \widehat{\phi}(k) e^{2\pi i k x}.$$

- $x = 0$ gives the PSF: $\sum_{k\in\mathbb{Z}^n} \phi(k) = \sum_{k\in\mathbb{Z}^n} \widehat{\phi}(k)$.

FT behavior under linear transformation

- Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is a non-singular linear operator. $u \in \mathcal{S}'$, $v = u \circ T$.
- Change of variables formula for integration implies

$$\widehat{v} = \frac{1}{|\det T|} \widehat{u} \circ T^{-\top}.$$

- Write $\mathbb{R}^n = H \oplus H^\perp$, H a linear subspace.
- Projection onto subspace defined by

$$(\pi_H f)(h) := \int_{H^\top} f(h+x)\,dx, \quad (h \in H).$$

- For $\xi \in H$: $\widehat{\pi_H f}(\xi) = \widehat{f}(\xi)$ (Fubini).

Analyticity of the FT

- Compact support: $f : \mathbb{R} \to \mathbb{C}$, $f \in L^1(\mathbb{R})$, $f(x) = 0$ for $|x| > R$.
- FT defined by
$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x}\, dx.$$
- Allow $\xi \in \mathbb{C}$, $\xi = s + it$ in the formula.
$$\widehat{f}(s+it) = \int f(x) e^{2\pi t x} e^{-2\pi i s x}\, dx$$
- Compact support implies $f(x)e^{2\pi t x} \in L^1(\mathbb{R})$, so integral is defined.
- Since $e^{-2\pi i x \xi}$ is analytic for all $\xi \in \mathbb{C}$, so is $\widehat{f}(\xi)$.
- Paley–Wiener: $f \in L^2(\mathbb{R})$. The following are equivalent:
  (a) f is the restriction on $\mathbb{R}$ of a function F holomorphic in the strip $\{z : |\Im z| < a\}$ which satisfies
$$\int |F(x+iy)|^2\, dx \le C, \quad (|y| < a)$$
  (b) $e^{a|\xi|}\widehat{f}(\xi) \in L^2(\mathbb{R})$.

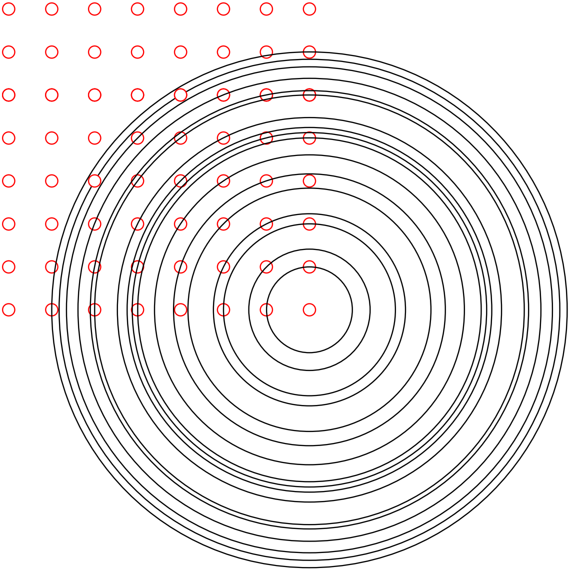
Application: the Steinhaus tiling problem

- Question of Steinhaus: Is there $E \subseteq \mathbb{R}^2$ such that no matter how translated and rotated it always contains exactly one point with integer coordinates?
- Two versions: E is required to be measurable or not
- Non-measurable version was answered in the affirmative by Jackson and Mauldin a few years ago.
- Measurable version remains open.
- Equivalent form (R_θ is rotation by θ):

$$\sum_{k \in \mathbb{Z}^2} \mathbf{1}_{R_\theta E}(t+k) = 1, \quad (0 \le \theta < 2\pi,\ t \in \mathbb{R}^2). \tag{5}$$

- We prove: there is no <u>bounded</u> measurable Steinhaus set.
- Integrating (5) for $t \in [0,1]^2$ we obtain $|E| = 1$.
- LHS of (5) is the $\mathbb{Z}^2$-periodization of $\mathbf{1}_{R_\theta E}$. Hence $\widehat{\mathbf{1}_{R_\theta E}}(k) = 0$, $k \in \mathbb{Z}^2 \setminus \{0\}$.
- $\widehat{\mathbf{1}_E}(\xi) = 0$, whenever ξ on a circle through a lattice point

The circles on which $\widehat{\mathbf{1}_E}$ must vanish



Application: the Steinhaus tiling problem: conclusion

- Consider the projection f of $\mathbf{1}_E$ on $\mathbb{R}$.
  E bounded $\Longrightarrow$ f has compact support, say in $[-B, B]$.
- For $\xi \in \mathbb{R}$ we have $\widehat{f}(\xi) = \widehat{\mathbf{1}_E}(\xi, 0)$, hence

$$\widehat{f}\left(\sqrt{m^2+n^2}\right) = 0, \quad (m,n) \in \mathbb{Z}^2 \setminus \{0\}.$$

- Landau: The number of integers up to x which are sums of two squares is $\sim Cx/\log^{1/2} x$.
- Hence $\widehat{f}$ has almost R^2 zeros from 0 to R.
- $\operatorname{supp} f \subseteq [-B, B]$ implies $\left|\widehat{f}(z)\right| \le \|f\|_1 e^{2\pi B|z|}$, $z \in \mathbb{C}$
- But such a function can only have $O(R)$ zeros from 0 to R.

Zeros of entire functions of exponential type

- JENSEN's formula: F analytic in the disk $\{|z| \le R\}$, z_k are the zeros of F in that disk. Then

$$\sum_k \log \frac{R}{|z_k|} = \int_0^1 \log \left| F(Re^{2\pi i\theta}) \right| d\theta.$$

- It follows

$$\#\{k:\ |z_k| \le R/e\} \le \int_0^1 \log \left| F(Re^{2\pi i\theta}) \right| d\theta$$

- Suppose $|F(z)| \le Ae^{B|z|}$. Then RHS above is $\le BR + \log A$.
- Such a function F can therefore have only $O(R)$ zeros in the disk $\{|z| \le R\}$.